

CONVEX DECOMPOSITIONS OF REAL PROJECTIVE SURFACES II: ADMISSIBLE DECOMPOSITIONS

SUHYOUNG CHOI

Dedicated to the memory of Professor D. S. Rim

Abstract

A real projective surface is a differentiable surface with an atlas of charts to real projective plane \mathbf{RP}^2 such that transition functions are restrictions of projective automorphisms of \mathbf{RP}^2 . Let Σ be an orientable compact real projective surface with convex boundary and negative Euler characteristic. Then Σ uniquely decomposes along mutually disjoint imbedded closed projective geodesics into compact subsurfaces that are maximal annuli, trivial annuli, or maximal purely convex real projective surfaces. This is a positive answer to a question by Thurston and Goldman raised around 1977.

We assume that surfaces in this paper are orientable always. Let S be a real projective surface with convex boundary. We say that S is the *sum* of subsurfaces S_1, \dots, S_n if S is the union of S_1, \dots, S_n , and if $S_i \cap S_j$ is the union of imbedded closed geodesics disjoint from one another or the empty set whenever i and j are integers satisfying $1 \leq i < j \leq n$ (compare with §3.1 of [14]). If S is the sum of S_1, \dots, S_n , then we say that S *decomposes* into S_1, \dots, S_n (*along closed geodesics*) and that $\{S_1, \dots, S_n\}$ is a *decomposition collection* of S . (See Appendix B of [16], and [26] for examples of this summation process.) This definition is slightly different from the one by Goldman [14] since we do not have the principal boundary conditions.

Let D be an arbitrary compact simply convex domain in a 2-dimensional sphere S^2 such that there is a segment α and a compact smooth arc β with two common endpoints p and q such that the boundary of D ∂D is $\alpha \cup \beta$. The quotient projective surface of $D - \{p, q\}$ by a properly discontinuous and free action of $\langle \vartheta \rangle$ for a hyperbolic or quasi-hyperbolic projective automorphism ϑ is called a *primitive trivial annulus*. (See [5])

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for the definitions of simple convexity and quotient projective surfaces.) It is a compact annulus with convex boundary. One of its boundary components is not geodesic, and the other is geodesic. A *trivial annulus* in S is a primitive trivial annulus A included in S such that the nongeodesic component of δA is a component of δS . (Clearly, a trivial annulus does not include two components of δS if the Euler characteristic of S is negative.) For example, given a hyperbolic projective surface R with convex boundary and an imbedded closed geodesic α in the interior R^0 of R freely homotopic to a component β of δR , the annulus with boundary $\alpha \cup \beta$ is a trivial annulus. In general, a compact convex projective surface with geodesic boundary can be extended to a compact projective surface with convex boundary. In this case, the difference between the extended surface and the original surface is often given by trivial annuli.

A *maximal annulus* in S is a compact annulus with geodesic boundary that is not a proper subset of a compact annulus with geodesic boundary in S .

A *purely convex surface* is a convex compact surface A with negative Euler characteristic that does not include a compact annulus with geodesic boundary freely homotopic to a component of δA or include a trivial annulus. A *maximal purely convex surface* in S is a purely convex surface in S that is not a proper subset of a purely convex surface in S .

We call trivial annuli, maximal annuli, and maximal purely convex surfaces in S *admissible subsurfaces* in S . We put the admissible subsurfaces of S into three different categories: (1) trivial annuli, (2) maximal annuli, (3) maximal purely convex surfaces. If S decomposes into admissible subsurfaces of S , then the decomposition collection is said to be *admissible*.

In this paper, we prove the admissible decomposition theorem using the main theorem of our previous paper [5]. We will use the notation and results of the paper [5].

Admissible Decomposition Theorem. *Let Σ be an orientable compact real projective surface with convex boundary and negative Euler characteristic. Then Σ admits a unique admissible decomposition collection.*

As we said in [5], this answers a question of Thurston and Goldman raised in 1977. A similar theorem is true when Σ is not orientable [7]. It is also claimed that if Σ is closed, then Σ decomposes into convex compact surfaces [6] (for some consequences, see [8]).

Let us outline the contents of this paper. In §1 we classify projective automorphisms of S^2 according to action and identify hyperbolic, quasi-

hyperbolic, and planar automorphisms. We classify elementary annuli, building blocks of annuli.

We next discuss three important geometric objects: tight curves, π -annuli, and convex surfaces. A tight curve is a closed geodesic in S , which lifts to a convex open line in the universal covering \tilde{S} . In §2, we show that tight curves have similar properties to those of closed geodesics in hyperbolic surfaces.

In §3, we discuss π -annuli, important objects in the article [5]. We classify π -annuli in terms of elementary annuli, and show that given a π -annulus with a projective map to S , the map is a finite covering map onto an imbedded π -annulus. Finally, we discuss the intersection properties of maximal annuli that include imbedded π -annuli.

In §4, we discuss properties of convex compact surfaces generalizing results of Kuiper [20]. First we show that the holonomy of each essential closed curve in S is hyperbolic or quasi-hyperbolic. Next, given an essential simple closed curve, we show that S includes an imbedded tight curve freely homotopic to it and that this is the unique one unless it is freely homotopic to a component of δS . We show that a convex compact surface decomposes into subsurfaces that are elementary annuli, trivial annuli, or purely convex surfaces. We end with discussing the intersection property of purely convex surfaces, annuli with geodesic boundary, and trivial annuli.

In §5, we give the proof of the admissible decomposition theorem. The idea of the proof is to collect all imbedded π -annuli in Σ and find maximal annuli that include them. We subtract these maximal annuli and trivial annuli from Σ . The main theorem of [5] implies by §3 that the closure of each component of the complement in Σ is a purely convex surface.

In Appendix A, we present various standard facts about curves in surfaces. Since we prove these by using hyperbolic structures, we separate this material out from the main text (see Casson and Bleiler [3]). We will follow the standard terminology of hyperbolic geometry in this section (see Maskit [21]).

In Appendix B, for our purposes in this paper, we present a proof of a version of the annulus decomposition theorem of Goldman [14].

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1. Projective automorphisms and elementary annuli

1.1. The one-dimensional sphere S^1 has a one-dimensional real projective structure induced from its double covering map to the one-dimensional real projective space RP^1 . (An *antipodal pair* in S^1 or S^2 is a pair of points antipodal to each other.) A projective automorphism of S^1 is induced by a unique element of the group $SL_{\pm}(2, \mathbf{R})$ of linear automorphisms of \mathbf{R}^2 of determinant ± 1 . Hence, the action of $\langle \vartheta \rangle$ where ϑ is a projective automorphism of S^1 preserving orientation and corresponding to a matrix with nonnegative eigenvalues can be described as one of the following:

(1) ϑ has four fixed points composing two antipodal pairs. One pair consists of attractors, and the other pair consists of repellers. The action is said to be *hyperbolic*.

(2) ϑ has an antipodal pair of fixed points. ϑ restricted to each component of the complement is a nontrivial affine translation if the component is given the natural affine structure. The action is said to be *parabolic*.

(3) ϑ has no fixed points, is an isometry of S^1 equipped with the standard metric, and is not the identity or the antipode map. The action is said to be *elliptic*.

(4) ϑ is the identity.

1.2. A projective automorphism of S^2 is induced by a unique element of the group $SL_{\pm}(3, \mathbf{R})$ of linear automorphisms of \mathbf{R}^3 of determinant ± 1 . Hence, the projective automorphism group $\text{Aut}(S^2)$ is isomorphic to $SL_{\pm}(3, \mathbf{R})$. Note that a projective automorphism is orientation preserving if and only if it corresponds to a matrix in $SL(3, \mathbf{R})$ (see [5]).

Let us classify projective automorphisms of S^2 . An element of $SL(3, \mathbf{R})$ is conjugate to exactly one of the following matrices where λ_1, λ_2 , and λ_3 are mutually distinct real numbers in $\mathbf{R} - \{0\}$:

$$(1) \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \lambda_1 \lambda_2 \lambda_3 = 1;$$

$$(2) \quad \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1^2 \lambda_2 = 1, \quad \lambda_1, \lambda_2 \in \mathbf{R} - \{1, 0, -1\};$$

$$(3) \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1^2 \lambda_2 = 1, \quad \lambda_1, \lambda_2 \in \mathbf{R} - \{1, 0, -1\};$$

$$(4) \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix};$$

$$(5) \quad \begin{bmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad u = \pm 1;$$

$$(6) \quad \begin{bmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad u = \pm 1;$$

$$(7) \quad \begin{bmatrix} r \cos(\theta) & -r \sin(\theta) & 0 \\ r \sin(\theta) & r \cos(\theta) & 0 \\ 0 & 0 & r^{-2} \end{bmatrix}, \quad r > 0, \quad 0 < \theta < 2\pi, \quad \theta \neq \pi.$$

We say that a matrix in $SL(3, \mathbf{R})$ is of type (i) if it is conjugate to a matrix (i) above, $i = (1), (2), \dots, (7)$. An orientation-preserving projective automorphism corresponding to a matrix of type (1) with positive eigenvalues is said to be *hyperbolic* (or *positive hyperbolic*). A projective automorphism corresponding to a matrix of type (2) with positive eigenvalues is said to be *quasi-hyperbolic*. A projective automorphism corresponding to a matrix of type (3) with positive eigenvalues is said to be *planar*. (Compare to Goldman [14], [16].)

1.3. Let us describe the fixed points and invariant great circles of orientation-preserving projective automorphisms corresponding to the above matrices when eigenvalues are nonnegative. (For each case, there are no other fixed points or invariant great circles other than what are described.)

(1) There are six fixed points, which compose three antipodal pairs. There is an antipodal pair of attractors, an antipodal pair of repellers, and an antipodal pair of points that are not attractors or repellers. Each great circle containing two fixed points not antipodal to each other is an

invariant great circle. There are three invariant great circles. The action on each of the great circles is hyperbolic.

(2) There are four fixed points which compose two antipodal pairs. There is an invariant great circle including only one of the pairs, the action on which is parabolic. The antipodal pair not in the great circle consists of attractors unless it consists of repellers. The only other invariant great circle passes through all fixed points. The action on it is hyperbolic.

(3) There is a great circle S^1 consisting of fixed points and an antipodal pair of fixed points not on S^1 . The antipodal pair consists of attractors unless it consists of repellers. Except for S^1 , a great circle is invariant if and only if it passes through the antipodal pair and an antipodal pair in S^1 . The action on each of the great circles is hyperbolic.

(4) There is an antipodal pair of fixed points and a unique invariant great circle, which includes the pair. The action on the great circle is parabolic.

(5) The set of fixed points composes a great circle S^1 . It includes an antipodal pair of points such that a great circle passes through them if and only if it is invariant. The action on each invariant great circle is parabolic whenever the great circle does not equal S^1 .

(6) The action is that of the identity map.

(7) The action has an invariant great circle and an antipodal pair of fixed points not on the great circle. The action on the great circle is elliptic. The pair consists of attractors if $r < 1$ and consists of repellers if $r > 1$. If $r = 1$, then each point of the pair is neither an attractor nor a repeller.

Let us state a convenient criterion to determine the type of a projective automorphism.

Lemma. *Let ϑ be an orientation-preserving projective automorphism of S^2 . Suppose that ϑ has two fixed points not antipodal to each other and that the action of $\langle \vartheta \rangle$ on the segment connecting the two points is not trivial. Then ϑ is hyperbolic, quasi-hyperbolic, or planar.*

Proof. Since there are two fixed points, at least two eigenvalues are positive; all eigenvalues are thus positive. ϑ does not correspond to a matrix of type (4) or (7) since the two fixed points of ϑ are not antipodal. If ϑ corresponds to a matrix of type (5) or (6), then the great circle containing the two fixed points is the set of fixed points. Hence, ϑ corresponds to a matrix of type (1), (2), or (3), and the lemma is proved.

1.4. We introduce the notion of elementary annuli (compare with §3 of [14] and [4]).

Let ϑ be an arbitrary hyperbolic projective automorphism. Then there are three noncollinear fixed points s , m , and w . The respective antipo-

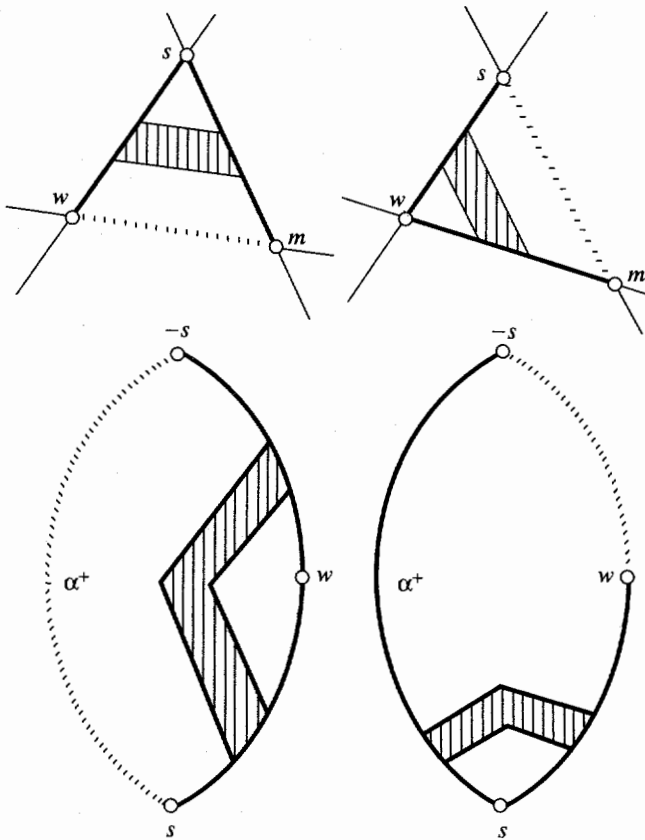


FIGURE 1. ELEMENTARY ANNULI.

dal points $-s$, $-m$, and $-w$ are also fixed. We assume that the respective eigenvalues of s , m , and w are strictly decreasing. Then there is an invariant closed triangle $\Delta(smw)$. The quotient projective surface of $\Delta(smw)^o$ under $\langle \vartheta \rangle$ is called an *open elementary annulus of type I*.

We may make this into compact annuli with geodesic boundary. The quotient surface of $\Delta(smw)^o \cup \overline{sw}^o \cup \overline{sm}^o$ under the action of $\langle \vartheta \rangle$ and the quotient surface of $\Delta(smw)^o \cup \overline{sw}^o \cup \overline{mw}^o$ under the action of $\langle \vartheta \rangle$ are called *elementary annuli of type I*. Projective annuli projectively homeomorphic to these are also called *elementary annuli of type I*. A boundary component of an elementary annulus of type I is said to be *strong* if it corresponds to \overline{sw}^o , and *weak* otherwise. See Figure 1.

As an aside, let us note that the respective interiors of the above two annuli are projectively homeomorphic but that they are not projectively

homeomorphic by an orientation preserving map. In general, they are not projectively homeomorphic by any map unless one of the eigenvalues of ϑ equals 1.

Let ϑ be a quasi-hyperbolic projective automorphism. ϑ has two fixed points s and w , where their respective antipodal points $-s$ and $-w$ are also fixed. Assume that the respective eigenvalues of s and w are strictly decreasing. Let S^1 be the great circle on which the action of ϑ is parabolic. Then either w or s belongs to S^1 . Suppose that $s \in S^1$. The great circle S^1 includes an invariant segment α^+ with endpoints s and $-s$ such that the sequence $\{\vartheta^n(x)\}$, $n = 1, 2, \dots$, converges to s for each point x of the open line $\alpha^{+,o}$. Let B be the lune bounded by α^+ , \overline{sw} , and $\overline{-sw}$. The quotient of B^o by the action of $\langle \vartheta \rangle$ is called an *open elementary annulus of type II* (Figure 1).

Let us consider the quotient surface of $B^o \cup \overline{sw}^o \cup \overline{-sw}^o$ under the action of $\langle \vartheta \rangle$. This is a compact projective annulus with geodesic boundary (a π -annulus in this case). We call it and a projective annulus projectively homeomorphic to it *elementary annuli of type IIa*.

The next is the quotient surface of $B^o \cup \alpha^{+,o} \cup \overline{sw}^o$ under the action of $\langle \vartheta \rangle$. This is again a compact projective annulus with geodesic boundary. We call it and a projective annulus projectively homeomorphic to it *elementary annuli of type IIb*. A boundary component of an elementary annulus of type IIa or IIb is said to be *strong* if the component corresponds to \overline{sw}^o or $\overline{-sw}^o$, and *weak* otherwise.

We remark that there is no projective homeomorphism between an elementary annulus of type I and an elementary annulus of type IIa or IIb. An elementary annulus of type IIa and an elementary annulus of type IIb are not projectively homeomorphic although their respective manifold interiors are.

2. Tight curves

Let S be a real projective surface with convex boundary. Let (dev, h) be the development pair of S , and let \tilde{S} be its universal cover with the covering map pr . A closed geodesic $\alpha: S^1 \rightarrow S$, where S^1 is a circle, is called a *tight curve* in S if its lift to \tilde{S} is a geodesic imbedding onto a convex open line. For example, geodesics realizing boundary component curves of elementary annuli are tight curves. A closed curve in a convex projective surface of negative Euler characteristic is a geodesic if and only if it is a tight curve. This may be verified by Lemma 1.5 of [5].

Given a closed curve α in S , it corresponds to an element $[\alpha]$ of the fundamental group $\pi_1(S)$ of S . If $h([\alpha])$ is hyperbolic (resp. quasi-hyperbolic, planar), then α is said to be *hyperbolic* (resp. *quasi-hyperbolic*, *planar*). This is well-defined, independently of the choices of the homotopy classes and development pairs of S . A principal closed geodesic in [16] is an example of a hyperbolic closed curve. See §4 for further examples of these curves. (For topological aspects of closed curves, refer to Appendix A.)

The main aim of this section is to show that tight curves in S behave almost like closed geodesics in hyperbolic surfaces (see [10]). We show that a tight curve freely homotopic to a simple closed curve is an imbedding onto an imbedded closed curve, that two imbedded tight curves intersect minimally, and that given a hyperbolic or quasi-hyperbolic closed curve, at most finitely many tight curves are freely homotopic to it. Next, we show that tight curves are not planar. Lastly, we discuss characterizing properties of elementary annuli and trivial annuli and show that S includes only finitely many trivial annuli.

2.1. Let us list a few basic properties of tight curves. Let $\alpha: \mathbf{S}^1 \rightarrow S$ be a tight curve. Then its lift $\tilde{\alpha}: \mathbf{R} \rightarrow \tilde{S}$ is injective. Thus α is essential. Next, Lemma 3.4 of [5] implies that if α passes through a point of δS , then α is a covering map onto an imbedded tight curve that is a component of δS . In other words, either α maps into S° or α is a covering map onto a component of δS . If β is an imbedded tight curve in S , then it follows that β is either a subset of S° or a component of δS .

2.2. We assume that S is homeomorphic to a compact annulus or a cover of a compact surface with negative Euler characteristic. We will be using a generic hyperbolic metric d on S obtained as in Appendix A. The induced metric on an arbitrary cover of S will also be denoted by d .

Proposition. *Let α and β be tight curves in S freely homotopic to simple closed curves.*

(1) *Suppose that $\gamma: \mathbf{S}^1 \rightarrow S$ is a tight curve freely homotopic to a finite covering map of an imbedded closed curve. Then γ is a finite covering map of an imbedded tight curve.*

(2) *α is an imbedding onto a one-dimensional submanifold of S ; so is β .*

(3) *The respective image submanifolds α_1 and β_1 of α and β either are identical or have minimal intersection.*

(4) *If α and β intersect at a point and intersect trivially in homotopy, then α_1 and β_1 are identical.*

(5) Suppose that α and β intersect trivially in homotopy. Then α_1 and β_1 are disjoint if α_1 is not freely homotopic to β_1 , and are disjoint or equal otherwise.

(6) Let $\gamma: \mathbf{S}^1 \rightarrow S$ be a tight curve. Then there is a finite cover S' of S such that γ lifts to a simple tight curve $\gamma': \mathbf{S}^1 \rightarrow S'$.

(7) Assume that S is compact. Let $\gamma: \mathbf{S}^1 \rightarrow S$ be a hyperbolic or quasi-hyperbolic closed curve. Then there are only finitely many tight curves in S freely homotopic to γ , whose images are distinct from one another.

Proof. (1) Suppose that two simply convex segments in S share two endpoints, and are respectively realized by maps homotopy equivalent with endpoints fixed. Then it is clear that they are identical.

Let $\tilde{\gamma}: \mathbf{R} \rightarrow \tilde{S}$ be a lift of γ , which is a geodesic in \tilde{S} and whose image is a convex maximal line, say l . A deck transformation ϑ corresponding to γ and $\tilde{\gamma}$ satisfies $\vartheta(l) = l$. Let φ be an arbitrary deck transformation. Suppose that $\varphi(l) \neq l$. Then $\varphi(l)$ and l are transversal. The first paragraph implies that l and $\varphi(l)$ may not meet at more than two points. Lemma 3 of Appendix A shows that $l \cap \varphi(l) = \emptyset$. Therefore, we have either $l = \varphi(l)$ or $l \cap \varphi(l) = \emptyset$ for each deck transformation φ . It follows that γ is a finite covering map onto an imbedded tight curve.

(2) By (1), α is a finite covering map onto an imbedded closed curve. Since S is orientable, α is an imbedding onto an imbedded closed curve.

(3) Suppose that $\alpha_1 \neq \beta_1$. Then α_1 and β_1 intersect transversally. The first paragraph of the proof of (1) and the definition of minimal intersection (see Casson and Bleiler [3]) imply that α_1 and β_1 intersect minimally. Hence, α_1 and β_1 either are identical or intersect minimally.

(4) By (3), α_1 and β_1 either are identical or have minimal intersection. Since the latter is not true, it follows that $\alpha_1 = \beta_1$.

(5) This follows from (4).

(6) There is a finite cover S' of S such that γ lifts to a closed curve $\gamma': \mathbf{S}^1 \rightarrow S'$ such that γ' is freely homotopic to a simple closed curve (see Scott [23], [24]). (2) implies the conclusion.

(7) Suppose not. We may assume without loss of generality that γ is a simple closed curve by lifting to a finite cover of S . Let $\{\gamma_i | i \in \mathbf{N}\}$ be a countable collection of tight curves freely homotopic to γ such that γ_j and γ_k are maps with distinct images whenever $j \neq k$. Parts (2) and (5) imply that γ_i for each i is a simple tight curve and that the images of γ_i are imbedded tight curves disjoint from one another.

Let ϑ be the deck transformation of \tilde{S} corresponding to γ . Let S' be the cover of S corresponding to ϑ ; that is, it is the quotient space of \tilde{S}

by the action of the group of deck transformations generated by ϑ . Then γ lifts to a simple closed curve γ' in S' . Hence, each γ_i lifts to a tight curve γ'_i in S' freely homotopic to γ' . By (2), γ'_i is simple. For each i , let γ''_i denote the image submanifold of γ'_i . Since every γ'_i is freely homotopic to γ' in S' , (5) implies that γ''_i and γ''_j are disjoint whenever $i \neq j$.

Let $\text{pr}'' : S' \rightarrow S$ be the covering map induced from the universal covering map $\text{pr} : \tilde{S} \rightarrow S$. pr'' is a local isometry. Since S'^o is homeomorphic to an open annulus, γ''_i and γ''_j for each $i, j > 1$, bound a unique annulus in S' . Let us denote it by A'_i . The above two paragraphs and a result of §6 of Appendix A show that $\text{pr}''|_{A'_i}$ is an imbedding and, hence, that $\{d\text{-area}(A'_i)\}, i > 1$, is a bounded sequence.

If the number of elements of $\{\gamma''_i\}$ intersecting each compact subset of S' is finite, then

$$\{d\text{-area}(A'_i)\} \rightarrow \infty \text{ as } i \rightarrow \infty.$$

Hence, $\{\gamma''_i\}$ is not locally finite.

This implies that S' contains a point p that is an accumulation point of $\bigcup_i \gamma''_i$ and is not a point of it. Let l_i denote the inverse image of γ''_i under the covering map from \tilde{S} to S' for each i . Let \tilde{p} denote a point of \tilde{S} corresponding to p , and let $B(\tilde{p})$ be a tiny disk of \tilde{p} (see §1.11 in [5]). Let us list obvious properties of l_i :

- (i) Each l_i is an open ϑ -invariant convex line.
- (ii) Infinitely many elements of $\{l_i\}$ intersect the interior of $B(\tilde{p})$.
- (iii) l_i and l_j are disjoint whenever $i \neq j$.

The above properties of l_i yield the following properties of $\text{dev}(l_i)$.

(i) For each i , $\text{dev}(l_i)$ is an $h(\vartheta)$ -invariant convex line, where h is the holonomy homomorphism.

(ii) Since $\text{dev}|_{B(\tilde{p})}$ is an imbedding, $\text{dev}(l_i)$ is distinct from $\text{dev}(l_j)$ whenever l_i and l_j intersect $B(\tilde{p})$ with $i \neq j$.

These contradict the fact that S^2 includes only finitely many $h(\vartheta)$ -invariant convex lines and proves (7).

2.3. We will now prove the following important property.

Proposition. *Suppose that S is compact and that $\chi(S) < 0$. If $\alpha : S^1 \rightarrow S$ is a tight curve in S , then α is not planar.*

Proof. Suppose that α is planar. We may assume without loss of generality that α is simple by lifting to a finite cover of S by Proposition 2.2(6). Let S' be the cover of S corresponding to α ; that is, S' is the quotient of \tilde{S} by the action of the group generated by a deck transformation ϑ corresponding to α and its lift $\tilde{\alpha} : \mathbf{R} \rightarrow \tilde{S}$. Let $\text{pr}' : \tilde{S} \rightarrow S'$

denote the covering map, and $\alpha': S^1 \rightarrow S'$ a lift of α . By Proposition 2.2(2), α' imbeds onto an imbedded tight curve α'' in S' .

Let A be the subset of points of S' through which a tight curve freely homotopic to α' passes. We claim that A is an open subset of S' . Let $x \in A$. Then x belongs to an imbedded tight curve α_x freely homotopic to α' . Let $\tilde{\alpha}_x$ be the image of a lift of α_x to \tilde{S} corresponding to ϑ . We have either $\tilde{\alpha}_x \subset \tilde{S}^o$ or $\tilde{\alpha}_x \subset \delta\tilde{S}$ by §2.1. In the first case, an open neighborhood \mathcal{U} of $\tilde{\alpha}_x$ satisfies the following conditions:

- $\text{pr}'|_{\mathcal{U}}$ is a covering map onto an annulus, an open neighborhood of α_x .
- $\text{dev}'|_{\mathcal{U}}$ is injective (thus, it is an imbedding).

The image $\text{dev}'(\mathcal{U})$ includes an $h(\vartheta)$ -invariant open triangle T including $\text{dev}'(\tilde{\alpha}_x)$ (see §1.3). Let $T' = (\text{dev}'|_{\mathcal{U}})^{-1}(T)$. The set T' is a ϑ -invariant open neighborhood of $\tilde{\alpha}_x$. Since $h(\vartheta)$ is planar, T' is foliated by ϑ -invariant convex lines. Thus, $\text{pr}'|_{T'}$ is a covering map onto an open annulus that is an open neighborhood of α_x and is foliated by imbedded tight curves freely homotopic to α_x . If we have $\tilde{\alpha}_x \subset \delta\tilde{S}$, then a similar argument shows that an open neighborhood of α_x is foliated by imbedded tight curves freely homotopic to α_x . Thus, A is open.

Let $\text{pr}'': S' \rightarrow S$ be the covering map induced from the universal covering map $\text{pr}: \tilde{S} \rightarrow S$. We claim that $\text{pr}''|_A$ is injective. Let x and y be two distinct points of A . Let α_x and α_y be the imbedded tight curves freely homotopic to α'' containing x and y respectively. Proposition 2.2(2) implies that $\text{pr}''|_{\alpha_x}$ and $\text{pr}''|_{\alpha_y}$ are injective maps onto imbedded tight curves. Thus, if $\alpha_x = \alpha_y$, then $\text{pr}''(x) \neq \text{pr}''(y)$. Suppose that $\alpha_x \neq \alpha_y$. By Lemma 5 of Appendix A, the respective images are distinct from each other. From Proposition 2.2(5) it follows that the respective images are disjoint, so that $\text{pr}''(x) \neq \text{pr}''(y)$. Thus, $\text{pr}''|_A$ is injective.

The injectivity of $\text{pr}''|_A$ yields that the d -area of A is finite. Let A' be the component of A including α'' . Clearly, A' is an annulus foliated by imbedded tight curves freely homotopic to α'' . We may assume without loss of generality that α'' is geodesic with respect to d (see Appendix A). Let us consider for each point x of α'' the maximal d -geodesic λ_x in A' perpendicular to α'' at x . Since the d -area of A' is finite, α'' contains a point x such that the d -length of the component μ_x of $\lambda_x \cap A'$ containing x is finite. There is a monotone sequence of points $\{p_i\}$ on a component of $\mu_x - \{x\}$ converging to the endpoint p of μ_x that is not x and satisfies $p \notin A$. Then for each i , p_i is an element of an imbedded tight curve

α_i freely homotopic to α'' . Let us denote by A_i the annulus bounded by α_i and α'' for each i . Clearly, $A_i \subset A'$. Let $\mu_{x,i}$ be the closure of the component of $\mu_x \cap A_i^o$ containing x for each i . Let q_i for each i be the endpoint of the d -geodesic segment $\mu_{x,i}$ that is not x . (See Figure 2.) Note that $\alpha'' \cap \mu_{x,i} = \{x\}$, $\alpha_i \cap \mu_{x,i} = \{q_i\}$, and $\mu_{x,i}^o \subset A_i^o$ hold for each i .

We have the following possibilities:

- (i) $\{d\text{-length}(\alpha_i)\}$ is a bounded sequence.
- (ii) $\{d\text{-length}(\alpha_i)\}$ is an unbounded sequence.

We will show below that each of these possibilities are contradictions. This will complete the proof.

(i) Assume without loss of generality that each α_i is parametrized by d -length. Then each α_i lies in a compact subset K of S' , α_i is a distance decreasing map, and, hence, $\{\alpha_i\}$ is bounded and uniformly continuous. Thus by the Ascoli-Arzelà theorem, a subsequence of $\{\alpha_i\}$ converges to a continuous closed curve $\alpha_\infty: S^1 \rightarrow S'$ passing through p . It is straightforward to show by a local argument that α_∞ is a tight curve and that α_∞ is freely homotopic to α'' . Hence $p \in A$, and this is a contradiction.

(ii) First, we cut open the annulus A_i along $\mu_{x,i}$ and lift it to \tilde{S} : To begin, we lift α . Let $\tilde{\alpha}^*$ be the image of the lift $\tilde{\alpha}$ of α to \tilde{S} . Recall that ϑ corresponds to $\tilde{\alpha}^*$ and α . Let x^* be a point of $\tilde{\alpha}^*$ corresponding to x . Then $\tilde{\alpha}^*$ includes a compact arc α^* with endpoints x^* and $\vartheta(x^*)$. Here $\text{pr}'|_{\alpha^*}$ maps onto α'' and is injective in the complement of the set of the endpoints of α^* . Now, we lift λ_x to a maximal d -geodesic λ_x^* that is d -perpendicular to α^* at x^* . A maximal d -geodesic $\vartheta(\lambda_x^*)$ is d -perpendicular to α^* at $\vartheta(x^*)$. The d -geodesic λ_x^* includes a compact arc $\mu_{x,i}^*$ with endpoints x^* and q_i^* , the point on λ_x^* corresponding to q_i . Note that $\text{pr}'|_{\mu_{x,i}^*}$ is an embedding onto $\mu_{x,i}$ and that so is $\text{pr}'|_{\vartheta(\mu_{x,i}^*)}$. There is also an arc α_i^* with endpoints q_i^* and $\vartheta(q_i^*)$ such that $\text{pr}'|_{\alpha_i^*}$ is onto α_i and is injective in the complement of the set of the endpoints. Since we have

$$\begin{aligned} \alpha^* \cap \mu_{x,i}^* &= \{x^*\} & \mu_{x,i}^* \cap \alpha_i^* &= \{q_i^*\}, \\ \alpha_i^* \cap \vartheta(\mu_{x,i}^*) &= \{\vartheta(q_i^*)\}, & \vartheta(\mu_{x,i}^*) \cap \alpha^* &= \{\vartheta(x^*)\}, \end{aligned}$$

the surface \tilde{S} includes a compact disk A_i^* bounded by four arcs α^* , $\mu_{x,i}^*$, α_i^* , and $\vartheta(\mu_{x,i}^*)$. It is clear that $\text{pr}'|_{A_i^*}$ maps onto A_i and is injective in the complement of $\mu_{x,i}^* \cup \vartheta(\mu_{x,i}^*)$ (see Figure 2, next page).

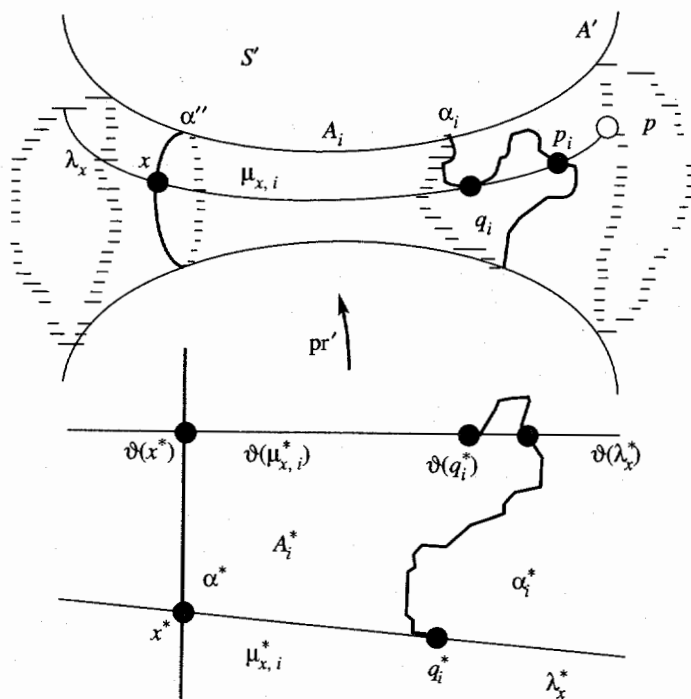


FIGURE 2

Let B_r denote the compact d -ball of radius r , $r > 0$, with center x^* . There are two possibilities:

- (α) There exists a positive constant r such that $\alpha_i^* \subset B_r$ for every i .
- (β) $\{\alpha_i^*\}$ is unbounded.

We will derive contradictions for these two cases. This will show that (ii) is a contradiction also.

(α) Cover B_r by a finite collection of open d -balls of radius $c_S/2$ where c_S is the constant of §2.4. Let N be the number of d -balls in the collection. For each i , let us choose $N+1$ points $r_{i,j}$, $j = 1, \dots, N+1$, on α_i^* so that the following properties hold:

- $r_{i,1} = q_i^*$, and $r_{i,N+1} = \vartheta(q_i^*)$.
- Let $\beta_{i,j}$ for each j , $j = 1, \dots, N$, be the arc in α_i^* with endpoints $r_{i,j}$ and $r_{i,j+1}$. Then $\beta_{i,j}^0$ does not contain $r_{i,k}$ for any k .
- The d -length of $\beta_{i,j}$ is independent of j .

We will now state two consequences that contradict each other: First, since $\{d\text{-length}(\tilde{\alpha}_i)\}$ is unbounded, it follows that $\{d\text{-length}(\beta_{i,j})\}$ for

each fixed j is unbounded. Second, let us fix i . The so-called pigeon-hole principle implies that there are at least two elements of $\{r_{i,j} | j = 1, \dots, N + 1\}$ in one of the covering d -balls. Let us say that they are $r_{i,k}$ and $r_{i,l}$ where $k < l$. Then by Lemma 2.4, the d -length of the arc on α_i^* connecting $r_{i,k}$ and $r_{i,l}$ is less than c'_S . Hence, $d\text{-length}(\beta_{i,k}) < c'_S$. This means that $d\text{-length}(\beta_{i,j}) < c'_S$ for every j . Since this inequality holds for every i , we have a contradiction.

(β) Since $\mu_{x,i} \subset \mu_x$ for every i , it follows that the d -distance from every point of $\mu_{x,i}^*$ or $\partial(\mu_{x,i}^*)$ to x^* is bounded above by a constant C independent of i . Let R be an arbitrary fixed real number greater than C ; let us choose an arbitrary fixed integer i for R such that α_i^* is not a subset of B_{3R} .

Let f be a function on \tilde{S} , that measures d -distances from x^* . For each r , $r > C$, we can divide A_i^* into three sets: $A^*(r)_{i,+}$, $A^*(r)_{i,0}$ and $A^*(r)_{i,-}$ consisting of points whose values under f are greater than, equal to, and less than r respectively. For a regular value r , $r > C$, of $f|_{\alpha_i^{*,0}}$, $A^*(r)_{i,0}$ is a one-dimensional submanifold in A_i^* , each component of which is a compact arc intersecting $\alpha_i^{*,0}$ precisely at the set of endpoints. Thus, we have constructed our first object: $A^*(r)_{i,0}$ on the level set $f^{-1}(r)$.

Let

$$\delta_R = \inf\{d\text{-length}(A^*(r)_{i,0}) | R < r < 2R, \\ r \text{ is a regular value of } f|_{\alpha_i^{*,0}}\},$$

and let

$$\varepsilon_R = \max\{\delta_R, d\text{-area}(A)/R\}.$$

Clearly, $\varepsilon_R > 0$. Let r' be a regular value of $f|_{\alpha_i^{*,0}}$ with $R < r' < 2R$ such that the d -length of $A^*(r')_{i,0}$ is less than $2\varepsilon_R$. Let x_i^* denote a point of $\alpha_i^* - B_{3R}$. Clearly, $x_i^* \in A^*(r')_{i,+}$, and $x^* \in A^*(r')_{i,-}$. There is a component arc ζ_i of $A^*(r')_{i,0}$ that separates x^* from x_i^* ; i.e., every path from x^* to x_i^* in A_i^* intersects ζ_i . Let n_i and m_i denote the endpoints of ζ_i in $\alpha_i^{*,0}$; let γ_i be the arc on α_i^* sharing endpoints n_i and m_i with ζ_i . Then $\gamma_i \ni x_i^*$. Thus, we have constructed our second and last objects for R : ζ_i and γ_i .

Since ζ_i is a subset of $A^*(r')_{0,i}$, the d -length of ζ_i is less than $2\varepsilon_R$. Since $n_i, m_i \in f^{-1}(r')$, and $x_i^* \notin B_{3R}$, the d -length of γ_i is greater than $2R$.

Hyperbolic geometry and the Sard theorem imply that the d -area of A_i^* is greater than or equal to $\delta_R R$. From the facts that $d\text{-area}(A)$ is finite, $A_i \subset A$, and pr' is a local isometry, it follows that $\delta_R \leq d\text{-area}(A)/R$. This means that $\varepsilon_R \rightarrow 0$ as $R \rightarrow \infty$, so that $d(n_i, m_i) \rightarrow 0$ and $d\text{-length}(\gamma_i) \rightarrow \infty$ as $R \rightarrow \infty$, which contradict Lemma 2.4, since γ_i is a convex segment.

2.4. In a real projective surface, a *tiny disk* is a compact disk simply convex with respect to the real projective structure (see §1.11 in [5]). Choose a covering of S by interiors of tiny disks and choose a finite subcovering. The finite subcovering corresponds to a locally finite covering \mathcal{O} of \tilde{S} by tiny disks. It follows that there is a positive constant c_S such that for every point p of \tilde{S} , the covering \mathcal{O} contains a tiny disk B of p such that $d(p, \text{bd } B) > c_S$.

Lemma. *Suppose that p and q are points of \tilde{S} connected by a convex segment α . Then there is a positive constant c'_S independent of p and q such that if $d(p, q) < c'_S$, then $d\text{-length}(\alpha) < c'_S$.*

Proof. Let B be a tiny disk belonging to \mathcal{O} . Given the Hausdorff metric associated with d , the collection of compact subsets of B forms a compact metric space. All segments and point sets in B form a closed subset of this metric space. (A subset is called a *point set* if it is the set consisting of a single point.) Let us call the subspace \mathcal{B} . If we assign the d -length of every point subset to be zero, then the real-valued function defined on \mathcal{B} assigning each element to its d -length is a continuous function. Hence, there is the supremum of the set of values of this function. The facts that for tiny disks mapping onto each other under deck transformations, the respective supremum are equal to each other and the covering \mathcal{O} is locally finite imply that there exists an upper bound of the collection of these supremums for tiny disks in \mathcal{O} . Let c'_S be this upper bound multiplied by two. This satisfies the conclusion of the lemma.

2.5. For the remainder of this section, let us discuss about facts on trivial annuli and elementary annuli. We call a compact annulus with convex boundary a *hyperbolic* (resp. *quasi-hyperbolic*) annulus if the boundary components are hyperbolic (resp. quasi-hyperbolic). Let us state a criterion for an annulus to be an elementary or trivial annulus.

Lemma. *Let A be a hyperbolic or quasi-hyperbolic annulus in S .*

(1) *Suppose that δA is a geodesic. Then A includes exactly two imbedded tight curves if and only if A is an elementary annulus.*

(2) *Suppose that the nongeodesic boundary components of δA are components of δS . Then a boundary component of δA is a unique imbedded*

tight curve in A if and only if A is a trivial annulus.

(3) An elementary or trivial annulus in S is not a proper subset of another elementary or trivial annulus in S .

Proof. (1), (2) These are straightforward consequences of §1 and the annulus decomposition theorem of Appendix B.

(3) This follows from (1) and (2).

2.6. Let us discuss trivial annuli in S . Let T_1 and T_2 be trivial annuli in S . Then their respective geodesic components α_1 and α_2 of δT_1 and δT_2 are imbedded tight curves. Since α_1 and α_2 are freely homotopic to components of δS , by Proposition 2.2, either α_1 and α_2 are disjoint or $\alpha_1 = \alpha_2$. The facts that a trivial annulus is the closure of a component of the complement of an imbedded tight curve, its geodesic boundary component, in S and $\chi(S) < 0$ and the above lemma imply that $T_1 = T_2$ or $T_1 \cap T_2 = \emptyset$. As a consequence, for each nongeodesic component of δS , there is at most one trivial annulus including the component. Therefore, S includes finitely many trivial annuli, which are mutually disjoint.

3. π -Annuli

3.1. Let M be a projective surface, and let \widetilde{M} be its universal cover. Let $D: \widetilde{M} \rightarrow S^2$ be a developing map for M , and let H be the holonomy homomorphism associated with D . A *strong tight curve* in M is a hyperbolic or quasi-hyperbolic tight curve $\alpha: S^1 \rightarrow M$ such that for its lift $\tilde{\alpha}: \mathbf{R} \rightarrow \widetilde{M}$, the map $D \circ \tilde{\alpha}$ is an imbedding onto a convex line connecting a fixed point of the largest eigenvalue with a fixed point of the smallest eigenvalue of $H(\vartheta)$ where ϑ is the deck transformation of \widetilde{M} corresponding to $\tilde{\alpha}$ and α . A *weak tight curve* is a tight curve in M that is not strong. For boundary components of elementary annuli, the definitions given in §1.4 agree with those given here. The geodesic boundary component of a trivial annulus is a strong tight curve. A component of δM is a principal closed geodesic if and only if it is a strong hyperbolic tight curve (see [16]).

These notions are invariant under projective maps and covering maps.

Lemma. Let $\alpha: S^1 \rightarrow M$ be a closed curve, and let $f: \mathcal{U} \rightarrow M'$ be a projective map defined on a neighborhood \mathcal{U} of the image of α mapping to a projective surface M' . Let β be a closed curve in M' , and let c_1 and c_2 be finite covering maps from S^1 to S^1 . Suppose that $f \circ \alpha \circ c_1$ is a reparametrization of $\beta \circ c_2$. Then α is a strong tight curve if and only if β is a strong tight curve.

Proof. Straightforward.

3.2. Let B be a great disk or a lune in S^2 . Let η_1 and η_2 be two great segments with common endpoints forming an antipodal pair and with $\eta_1 \cup \eta_2 = \delta B$. The quotient projective surface A of $(B^\circ \cup \eta_1^\circ) - \{x\}$ for a point x of η_1° under the action of $\langle \vartheta \rangle$ for a projective automorphism ϑ in $\text{Aut}(S^2)$ is a π -annulus (see [5]).

The projective automorphism ϑ has two invariant segments η_1 and η_2 and a fixed point x . Since ϑ is orientation-preserving, the endpoints of η_1 and η_2 are also fixed points. Lemma 1.3 implies that ϑ is hyperbolic, quasi-hyperbolic, or planar.

Suppose that S is compact, and that there is a projective map $f: A \rightarrow S$. Then $f|_\alpha$ for a component α of δA is a tight curve in S . Thus A must be hyperbolic or quasi-hyperbolic by Proposition 2.3.

Suppose that A is hyperbolic. It is clear that B is the union of two invariant triangles. It follows that A is the sum of two elementary annuli of type I. Suppose that A is quasi-hyperbolic. Then B is either an invariant lune or the union of two invariant lunes. In the first case, A is an elementary annulus of type IIa; in the second case, A is the sum of two elementary annuli of type IIb.

Let us observe that if A is a hyperbolic π -annulus, then the components of δA are either both strong or both weak. Moreover, A° includes a unique imbedded tight curve, which is weak if the components of δA are strong and is strong otherwise. If A is a π -annulus that is an elementary annulus of type IIa, then the components of δA are strong. In this case, A° includes no imbedded tight curve. If A is a π -annulus that is the sum of two elementary annuli of type IIb, then the components of δA are strong. Moreover, A° includes a unique tight curve, which is weak.

3.3. The aim of this section is to prove the following proposition:

Proposition. *Let $f: A \rightarrow S$ be a projective map. Then S includes an imbedded π -annulus F such that $f = i_F \circ c_A$ for an inclusion map $i_F: F \rightarrow S$ and a finite covering map $c_A: A \rightarrow F$; that is, the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{f} & S \\ c_A \downarrow & & \parallel \\ F & \xrightarrow{i_F} & S. \end{array}$$

3.4. A consequence of Proposition 3.3 and the main theorem in [5] is as follows (this was claimed in the introduction in [5]).

Theorem. *S is convex if and only if S does not include an imbedded π -annulus.*

Proof. Suppose that S is not convex. Then there is a π -annulus with a projective map to S by the main theorem [5]. Thus the above proposition shows that S includes an imbedded π -annulus. The converse portion is a consequence of the following lemma.

3.5. Lemma. *Let M be a real projective surface. Suppose that M includes a π -annulus E . Then M is not convex.*

Proof. Suppose that M is convex. Let (D, H) be a development pair of M . Since \tilde{M} is tame, D is an imbedding onto a convex domain Ω in S^2 by §1.4 in [5]. Let $p: \tilde{M} \rightarrow M$ be the universal cover, and let \tilde{E} be a component of $p^{-1}(E)$. There exists a deck transformation φ of \tilde{M} acting on \tilde{E} , and $D|\tilde{E}$ is an imbedding onto $(K^o \cup \zeta^o) - \{y\}$ for a lune or great disk K , a convex segment ζ in δK , and a point y of ζ^o . Since Ω is convex, and $(K^o \cup \zeta^o) - \{y\} \subset \Omega$, it follows that $y \in \Omega$. However, y is a fixed point of $H(\varphi)$. Since the action of the holonomy group on Ω is free, this is a contradiction. Thus, M is not convex.

3.6. We need some preliminary material before proving Proposition 3.3. Let M be a compact real projective surface with convex boundary. A *geodesic complex* K in M is a compact subset with the following property: for each point p of K , the surface M includes an open neighborhood \mathcal{U} of p such that

$$\mathcal{U} \cap K = \bigcup_{i=1}^n l_i$$

holds where each l_i , $i = 1, \dots, n$, is a maximal line in \mathcal{U} passing through p . For example, the union of finitely many imbedded closed geodesics and maximal geodesic segments in M is a geodesic complex. Also, the image of every closed geodesic in M is a geodesic complex.

Let K be a geodesic complex. A *regular point* of K is a point of K with a neighborhood in K that is a line, a *vertex* of K is a point of K that is not regular, and a *regular arc* of K is a component arc of K removed the set of vertices. Regular arcs of K are imbedded geodesics.

Let us state properties of K . (Let d' denote a complete metric on M). First, it is easy to see that there are finitely many vertices and regular arcs in K . Second, let $\gamma: [a, b] \rightarrow K$, $a < b$, be a d' -length parametrized geodesic where $\gamma(a)$ or $\gamma(b)$ does not belong to δM . Then the local condition of K implies that there is a d' -length parametrized geodesic $\gamma': I' \rightarrow K$, where I' is an interval including $[a, b]$ properly, and $\gamma'|[a, b] = \gamma$; that is, a d' -length parametrized geodesic γ' extends γ .

Lastly, we claim that one of the following holds for each point p of $K \cap M^o$:

- There is a closed geodesic in $K \cap M^o$ passing through p .
- There is a geodesic path $\gamma: I \rightarrow K$ for a compact interval I such that $\gamma(\delta I) \subset \delta M$, and γ passes through p .

Let us prove this. Choose a d' -length parametrized geodesic $\eta: J \rightarrow K$ for an interval J passing through p . Let $\eta': J' \rightarrow K$ for an interval J' be a maximal geodesic extending η ; that is, we assume that η' is a d' -length parametrized geodesic, that J is a subset of J' , that $\eta'|_J = \eta$, and that there is no d' -length parametrized geodesic extending η' . The interval J' either is unbounded or is compact. If J' is unbounded, then the compactness of K implies that there exists a point of a regular arc that η' passes and then passes again in the direction of the previous visit. This implies that η' passes p and then passes p again in the direction of the previous visit. Hence, η' corresponds to a closed geodesic η'' passing through p . Since η'' passes p , it follows that the image of η'' is a subset of M^o by Lemma 34 in [5]. If J' is compact, then $\eta'(\delta J') \subset \delta M$. Otherwise, η' is not maximal by the above paragraph. Thus the claim is proved.

3.7. We now begin the proof of Proposition 3.3. Let α and β be components of δA . As a first step, we will prove that $f|_\alpha$ is a covering map onto an imbedded tight curve in S . If $f(\alpha)$ intersects δS , then $f(\alpha)$ is a component of δS by §2.1. Hence, the claim is true in this case.

Suppose that $f(\alpha) \subset S^o$. Then the image $f(\alpha)$ is a geodesic complex. Since for each point of A , there is a neighborhood such that f restricted to it is a diffeomorphism onto a simply convex open disk or a disk projectively homeomorphic to a simply convex open disk in S^2 intersected with a closed hemisphere, $f^{-1}(f(\alpha))$ is a geodesic complex in A .

Let $L = f^{-1}(f(\alpha))$. By §3.6, there are the following possibilities:

- (1) There is a closed geodesic in $L \cap A^o$.
- (2) There is a geodesic $\gamma: I \rightarrow A$ for an interval I passing through a point of A^o such that $\gamma(\delta I) \subset \delta A$.
- (3) $L \subset \delta A$.

In case (1), Proposition 2.2(1) implies that the closed geodesic is a covering map onto an imbedded tight curve η in A^o . Let $\eta': S^1 \rightarrow A^o$ be a simple tight curve realizing η , and let $\alpha': S^1 \rightarrow A$ be a tight curve realizing α . Since η' is a closed geodesic, the image of $f \circ \eta'$ is $f(\alpha)$. This implies that $f \circ \eta' \circ c$ is a reparametrization of $f \circ \alpha' \circ c'$ for finite covering maps c and $c': S^1 \rightarrow S^1$. Thus, if α is strong, then η is also strong by

Lemma 3.1. However, this contradicts §3.2. If α is weak, then η is also weak; this is also a contradiction.

In case (2), the geodesic corresponds to a path in $(B^o \cup \eta^o) - \{x\}$ connecting two points of $\eta^o - \{x\}$ and passing through a point of B^o (see §3.2). Such a path cannot be geodesic, and this is a contradiction.

Since cases (1) and (2) do not occur, $L \subset \delta A$. In this case, L is either a component of δA or δA itself, and every point of L is regular. Thus $f(\alpha)$ is an imbedded tight curve, and $f|_\alpha$ is a finite covering map onto $f(\alpha)$, which we claimed. Similarly, it follows that $f|_\beta$ is a finite covering map onto an imbedded tight curve $f(\beta)$.

Let S' be the cover of S corresponding to $f(\alpha)$. The map f lifts to a projective map $f': A \rightarrow S'$. Proposition 2.2 implies that $f'|_\alpha$ and $f'|_\beta$ are finite covering maps onto the imbedded tight curves $f'(\alpha)$ and $f'(\beta)$ respectively.

Let H be the continuous function defined on S' that measures d -distances from $f'(\alpha)$. Since f' is an immersion, $H \circ f'$ achieves the maximum value at points of δA . This implies that $f'(\alpha) \neq f'(\beta)$. Since $f'(\alpha) \cap f'(\beta) = \emptyset$ by Proposition 2.2, $f'(\alpha)$ and $f'(\beta)$ bound a unique compact annulus, say A' , in S' . Using H , we can show similarly that the image of f' lies in A' . In fact, $f'(A^o) \subset A'^o$.

Since $f'(\alpha) \neq f'(\beta)$, Lemma 5 of Appendix A implies that $f(\alpha) \neq f(\beta)$. Proposition 2.2 implies that $f(\alpha) \cap f(\beta) = \emptyset$. Hence, $f(\alpha)$ and $f(\beta)$ bound a unique compact annulus F . Section 6 of Appendix A shows that the covering map from S' to S restricted to A' is an imbedding onto F . Hence, the image of f lies in F . Let $c_A: A \rightarrow F$ be the immersion such that $f = i_F \circ c_A$ for the inclusion map $i_F: F \rightarrow S$. Each of $c_A|_\alpha$ or $c_A|_\beta$ is a covering map onto an appropriate boundary component of F . Since A is compact, the inverse image of each point of F under c_A is a finite set. It is clear that $c_A(x) \in \delta F$ if and only if $x \in \delta A$. These imply that c_A is a covering map. By the definition of π -annuli, an orientable quotient projective surface of a π -annulus is a π -annulus. Thus, F is a π -annulus. This completes the proof of Proposition 3.3.

3.8. Let us now discuss imbedded π -annuli and maximal annuli including them. We need the following lemma.

Lemma. *Let A be an imbedded π -annulus in S . Let $\alpha: S^1 \rightarrow S$ be a tight curve. Then α intersects trivially in homotopy with every closed curve realizing a component of δA . Moreover, if α is simple, then we have either $\alpha_1 = \beta$ or $\alpha_1 \cap \beta = \emptyset$ for an arbitrary component β of δA and the image α_1 of α .*

Proof. Suppose that α intersects essentially with a component β of δA . By §3.6, α_1 is a geodesic complex. Since $\alpha_1 \cap A^o \neq \emptyset$ and δA is geodesic, $\alpha_1 \cap A$ is a geodesic complex in A . Since no regular arc of α_1 is tangent to β and $\alpha_1 \cap \beta \neq \emptyset$, a regular arc of $\alpha_1 \cap A$ ends at a point of β transversally. Let p be a point of A^o on the regular arc. Section 3.6 implies that there is a geodesic $\gamma: I \rightarrow A$ passing through p such that $\gamma(\delta I) \subset \delta A$. This is a contradiction as in the case (2) of §3.7. The rest of the conclusion follows from Proposition 2.2.

3.9. We claim that S can include only finitely many π -annuli. Proposition 2.2(7) and the fact that an imbedded π -annulus is the closure of a component of the complement of two imbedded tight curves imply that there are only finitely many imbedded π -annuli freely homotopic to a given simple closed curve. If two π -annuli A and B are not freely homotopic to each other, then each component of δA is disjoint from the components of δB by the above lemma, and it follows that A and B are disjoint. Since given a compact surface, every collection of imbedded essential closed curves disjoint from one another and not freely homotopic to one another has finitely many elements, S includes only finitely many imbedded π -annuli.

3.10. We now discuss maximal annuli. Given an arbitrary hyperbolic or quasi-hyperbolic annulus A with geodesic boundary, it is a subset of a unique maximal annulus by Proposition 2.2(7) and the annulus decomposition theorem of Appendix B. We restrict our study to a particular class of maximal annuli. We denote by $M(S)$ the collection of maximal annuli that are either freely homotopic to imbedded π -annuli or freely homotopic to the components of δS that have hyperbolic or quasi-hyperbolic holonomy. Note that if the above A is freely homotopic to a component of δS , then A is a subset of an element of $M(S)$. By Proposition 2.2 and Lemma 3.8, we obtain the next lemma.

Lemma. *Each imbedded π -annulus is a subset of an element of $M(S)$, for every distinct elements a and a' of $M(S)$, the annuli a and a' are not freely homotopic and satisfy $a \cap a' = \emptyset$, and $M(S)$ is finite.*

4. Convex surfaces

4.1. Suppose that S is convex and compact. Then S is projectively homeomorphic to the quotient projective surface Ω/Γ for a convex domain Ω of S^2 and a projective automorphism group Γ acting properly discontinuously and freely on Ω by Lemma 1.5 of [5]. Moreover, $\text{Cl}(\Omega)$

is a simply convex disk in S^2 , including no antipodal pair. Let us identify Ω with \tilde{S} in this section; this identifies Γ with the deck transformation group of \tilde{S} . (For examples of convex surfaces, see Goldman [16].)

It is well known that $Cl(\Omega)$ is not a triangle, a simply convex disk bounded by the union of three segments (see [1], [15], [17], [25]). Suppose not. Then there is a homomorphism from Γ to the permutation group of the three vertices of the triangle, and the kernel of this homomorphism is abelian since the vertices are fixed under the action of the kernel. Since $\pi_1(S)$ is not virtually abelian, this is a contradiction.

4.2. Suppose that ϑ is a hyperbolic automorphism in Γ . Then $Cl(\Omega)$ contains a unique attractor and a unique repeller. Let us denote the points by s and w respectively. $Cl(\Omega)$ contains at most one fixed point that is not an attractor or a repeller. Let us denote by m the third fixed point if it exists. Hence, \overline{sw} is a unique invariant segment in $Cl(\Omega)$ if m does not exist. \overline{sw} , \overline{sm} , and \overline{mw} are all the invariant segments in $Cl(\Omega)$ if m exists. They bound a unique invariant triangle $\Delta(smw)$ in $Cl(\Omega)$. Suppose that φ is a quasi-hyperbolic element of Γ . Then, similarly, $Cl(\Omega)$ contains a unique fixed point of the largest eigenvalue and a unique fixed point of the smallest eigenvalue. Let us denote them by h and l respectively. Then the unique φ -invariant segment in $Cl(\Omega)$ is \overline{hl} .

We will need the following preliminary lemma, on the properties of invariant segments. Note that the proof of (3) differs from that of Kuiper [20]. However, the lemma itself may be proved along Kuiper's argument.

Lemma. (1) *If m exists, then \overline{sm} , $\overline{mw} \subset bd \Omega$.*

(2) $\overline{hl} \subset bd \Omega$.

(3) *Each nontrivial element γ of Γ does not correspond to a matrix of type (4).*

(4) $\overline{sw}^o \subset \Omega$.

(5) *If m exists, then one of \overline{sm}^o and \overline{mw}^o is a subset of Ω , and the other is disjoint from Ω .*

(6) $\overline{hl}^o \subset \Omega$.

Proof. (1) Suppose that $\overline{sm}^o \cap \Omega^o \neq \emptyset$. Let p be a point of this intersection. Let S^1 be the ϑ -invariant great circle including \overline{sm} . Let U be an open disk in Ω containing p such that $U - S^1$ has two components. Choose a point x from a component of $U - S^1$, and choose a point y from the other component. Then $\{\vartheta^{-n}(x)\}$ converges to a repeller of ϑ , and $\{\vartheta^{-n}(y)\}$ converges to the antipodal point of the repeller. These points are in $Cl(\Omega)$. This contradicts the fact that Ω is simply convex.

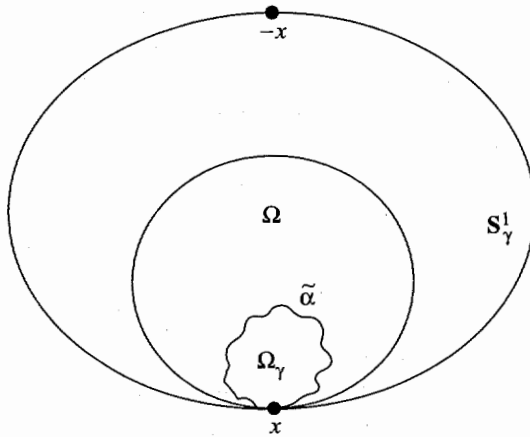


FIGURE 3. THE ACTION ON Ω OF A PROJECTIVE AUTOMORPHISM CORRESPONDING TO A MATRIX OF TYPE (4).

Hence, $\overline{sm} \subset \text{bd } \Omega$. It follows similarly that $\overline{mw} \subset \text{bd } \Omega$.

(2) A similar argument also shows that $\overline{hl} \subset \text{bd } \Omega$.

(3) By lifting to a finite cover of S , if necessary, we may assume without loss of generality that γ corresponds to an imbedded closed curve α in S^0 . Then \tilde{S} includes a simple component arc $\tilde{\alpha}$ of the inverse image of α that is γ -invariant.

Suppose that γ corresponds to a matrix of type (4). The automorphism γ has an invariant great circle S^1_γ and two fixed points, which are on S^1_γ , antipodal to each other. Since $\text{Cl}(\Omega)$ is γ -invariant, it follows that $\text{bd } \Omega$ contains a fixed point x of γ ; hence, the fixed points of γ are x and $-x$. The component of $S^2 - S^1_\gamma$ including Ω^o is such that for each compact subset A of it, the sequence $\{\gamma^n(A)\}$ converges to $\{x\}$ as $n \rightarrow \infty$. Moreover, $\{\gamma^{-n}(A)\}$ also converges to $\{x\}$ as $n \rightarrow \infty$ (see Figure 3). Hence, $\tilde{\alpha}$ is a simple curve starting and ending at x such that $\tilde{\alpha} \cup \{x\}$ is an imbedded closed curve (not necessarily differentiable).

It follows that Ω includes the unique open disk Ω_γ that is a component of $\Omega - \tilde{\alpha}$ and whose topological boundary set in S^2 equals $\tilde{\alpha} \cup \{x\}$. By Lemma 4 of Appendix A, Γ includes an element τ that sends a point $\tilde{\alpha}$ into Ω_γ and does not commute with γ . Since α is simple, $\tau(\tilde{\alpha})$ is disjoint from $\tilde{\alpha}$ and, hence, is a subset of Ω_γ . The curve $\tau(\tilde{\alpha})$ is a simple curve starting and ending at $\tau(x)$. Since $\tau(\tilde{\alpha})$ is infinitely long under the metric d of \tilde{S} (see §2.2 and Appendix A), $\tau(\tilde{\alpha})$ must end at x also. Thus $\tau(x) = x$. Thus, $\tau \circ \gamma \circ \tau^{-1}$ fixes x and corresponds to a matrix of

type (4). S_γ^1 is the unique supporting great circle of $Cl(\Omega)$ at x . Since $\tau(S_\gamma^1)$ is also a supporting great circle of $Cl(\Omega)$ at x , we have $S_\gamma^1 = \tau(S_\gamma^1)$. This means that γ and $\tau \circ \gamma \circ \tau^{-1}$ commute since they are both of type (4) and share an invariant great circle and a fixed point. This contradicts the fact that γ and τ do not commute (see §2 of Appendix A).

(4) The segment \overline{sw} is a subset of $Cl(\Omega)$. If $\overline{sw}^o \cap \Omega^o \neq \emptyset$, then $\overline{sw}^o \subset \Omega^o$. Suppose that $\overline{sw} \subset \text{bd } \Omega$ and that $\overline{sw} \subset Cl(\Omega) - \Omega$. As in (3), we assume without loss of generality that ϑ corresponds to an imbedded closed curve α . Let $\tilde{\alpha}$ be the simple component arc of the inverse image of α in \tilde{A} , which is ϑ -invariant. A description of the action of $\langle \vartheta \rangle$ implies that $\tilde{\alpha}$ is a simple arc with endpoints s and w and that $\tilde{\alpha} \cup \overline{sw}$ is the topological boundary in S^2 of an open disk Ω_ϑ that is a component of $\Omega - \tilde{\alpha}$. As before, we obtain a deck transformation τ that maps a point of $\tilde{\alpha}$ into Ω_ϑ and that does not commute with ϑ . Now an argument similar to one in (3) shows that this is a contradiction. Hence, $\overline{sw} \cap \Omega \neq \emptyset$. Let β be a component of $\overline{sw}^o \cap \Omega$. Since each component of $\overline{sw}^o \cap \Omega$ is the image of a lift of a component curve of δS , it follows that β is invariant under an element of Γ . Call it τ' . Then the action of $\langle \tau' \rangle$ on \overline{sw} cannot be nontrivial unless $\beta = \overline{sw}^o$ (see §7.1 of [5]). Hence, $\overline{sw}^o \subset \Omega$.

(5) Similarly to the proof of (4), it follows that $(\overline{sm}^o \cup \overline{mw}^o) \cap \Omega$ is not empty. Assume without loss of generality that $\overline{sm}^o \cap \Omega$ is not empty. Then similarly to the proof of (4), $\overline{sm}^o \subset \Omega$. Since the action of $\langle \vartheta \rangle$ on Ω is properly discontinuous, \overline{mw}^o is disjoint from Ω . Hence (5) is proved.

(6) This can be proved similarly to (4).

4.3. We present the proof of the following generalization of the work of Kuiper [20].

Lemma. *Let γ be an essential closed curve in S . Then γ is hyperbolic or quasi-hyperbolic.*

Proof. Let ϑ be an element of Γ corresponding to γ . Since ϑ is orientation preserving, ϑ corresponds to a matrix T belonging to $SL(3, \mathbf{R})$. Suppose that T is of type (1), (2), or (3) and that the eigenvalues of T are positive. Then Proposition 2.3 implies that ϑ is hyperbolic or quasi-hyperbolic.

We will show that no other possibility for T can happen. From the simple convexity of $Cl(\Omega)$ it follows easily that T is not of type (1), (2), or (3) with negative eigenvalues.

The Brouwer fixed-point theorem implies that $Cl(\Omega)$ contains at least one fixed point of ϑ in $\text{bd } \Omega$. Let us choose one, which we denote by p .

Let G be the set of supporting great circles at p . It is clear that G forms a topological space homeomorphic to a segment or is the set consisting of one element. The action of $\langle \vartheta \rangle$ induces an action on G . Hence, an element of G is fixed under ϑ by the Brouwer fixed-point theorem. The corresponding supporting great circle is ϑ -invariant. If T is of type (7), then no great circle passing through the fixed points of ϑ is ϑ -invariant. Hence, T is not of type (7).

Since Γ is isomorphic to $\pi_1(S)$, it contains no elements of order two. If T is of type (6), then $T^2 = \text{Id}$. Hence, T is not of type (6).

Suppose that T is of type (5). Let η be an arbitrary segment with endpoints p and $-p$ passing through a point r of Ω^0 . Then $\eta \cap \text{Cl}(\Omega)$ is a simply convex segment with two endpoints, one of which is p . The other endpoint is not $-p$. Let it be called q . Then q is a fixed point of ϑ^2 since η is a ϑ^2 -invariant segment. Since q is fixed, every point of η is a fixed point of ϑ^2 . Hence, r is fixed. This contradicts the fact that $\langle \vartheta^2 \rangle$ acts freely on Ω .

By Lemma 4.2, T is not of type (4). The proof is completed.

4.4. A *purely convex surface* is a convex compact surface M with negative Euler characteristic that does not include a compact annulus with geodesic boundary freely homotopic to a component of δM or include a trivial annulus. Since every principal closed geodesic is an imbedded strong tight curve, Proposition 4.5(8) implies that convex compact surfaces with principal geodesic boundary are examples of purely convex surfaces (see Goldman [16]).

Many convex surfaces are not purely convex. By a summation method described in §3.7 of Goldman [16], we may obtain a compact surface M_1 with geodesic boundary, which decomposes into a convex compact surface with principal geodesic boundary and an elementary annulus of type I. M_1 is convex by Lemma 5.4.

Note that one can obtain a convex surface with a boundary component that is a quasi-hyperbolic tight curve using Goldman's techniques [16]. There is a convex real projective structure on a compact pair-of-pants such that one of the boundary invariants equals (λ, τ) for every real numbers λ and τ satisfying $0 < \lambda < 1$, $\tau = 2/\sqrt{\lambda}$. This can be obtained by a slight extension of Theorem 4.1 of [16]. (The proof of the remark is to be supplied in another paper.) Since the boundary invariant is as above, the boundary component is not hyperbolic but quasi-hyperbolic by Lemma 4.3.

4.5. A *boundary elementary annulus* in S is an elementary annulus including a component of δS . The following proposition is a generaliza-

tion of the results of Kuiper [20] that every essential simple closed curve in a closed convex surface with negative Euler characteristic is freely homotopic to a closed geodesic.

Proposition. *Let α be an imbedded essential closed curve in S , and let ϑ be an element of Γ corresponding to α .*

(1) *If $\text{bd}\Omega$ includes a ϑ -invariant segment, then α is freely homotopic to a tight-curve component of δS , which corresponds to a ϑ -invariant open line in $\delta\Omega$.*

(2) *If α is hyperbolic and is freely homotopic to no component of δS , then S^o includes a unique imbedded tight curve freely homotopic to α , which is strong.*

(3) *If α is hyperbolic and is freely homotopic to a component of δS , then S either*

(i) *includes a unique imbedded tight curve freely homotopic to α , which is strong, or*

(ii) *includes two imbedded tight curves freely homotopic to α , where one is strong and lies in S^o (the other is weak and is a component of δS).*

(4) *If α is quasi-hyperbolic, then S includes a unique imbedded tight curve freely homotopic to α , which is strong and is a component of δS .*

(5) *α is hyperbolic if α is not freely homotopic to a component of δS .*

(6) *S includes a unique imbedded strong tight curve freely homotopic to α and at most one imbedded weak tight curve freely homotopic to α , which must be a component of δS .*

(7) *If S is purely convex, then S includes a unique imbedded tight curve freely homotopic to α , which is strong.*

(8) *Every component of δS is an imbedded strong tight curve if and only if S is purely convex.*

(9) *S is the sum of a purely convex surface and subsurfaces that are trivial annuli or boundary elementary annuli. Every two distinct elements of the collection of the trivial annuli and the boundary elementary annuli are disjoint. Each component β of δS is a strong tight curve if and only if β is a boundary component of the purely convex surface but is not that of the trivial annuli or the boundary elementary annuli.*

Proof. (1) Let us denote the endpoints of the invariant segment by p and q which are fixed points of ϑ . Suppose that ϑ is hyperbolic and that one of p or q is not an attractor or a repeller of ϑ . We assume without loss of generality that p is a repeller and that q is not an attractor. Then $\text{Cl}(\Omega)$ includes an attractor r . Lemma 4.2(1) implies that two invariant segments \overline{pq} and \overline{qr} are subsets of $\text{bd}\Omega$. The interior η of one of these

segments is a subset of Ω by Lemma 4.2(5). Since η is ϑ -invariant, under the quotient map from Ω , the open line η corresponds to an imbedded tight curve that is a component of δS and to which α is freely homotopic.

Suppose now that ϑ is hyperbolic and that each of p and q is an attractor or a repeller of ϑ . Then similarly to above, Lemma 4.2(4) implies that \overline{pq}^o is a subset of Ω and corresponds to a tight curve that is a component of δS and to which α is freely homotopic. If ϑ is quasi-hyperbolic, then the desired conclusion follows similarly.

(2) Since ϑ is hyperbolic, $\text{bd}\Omega$ contains an attractor and a repeller of $\langle\vartheta\rangle$. Let them be denoted by p and q respectively. Since α is not freely homotopic to a boundary component, \overline{pq} is not a subset of $\text{bd}\Omega$ by (1). Hence, $\overline{pq}^o \subset \Omega^o$. This line corresponds to an imbedded strong tight curve in S^o freely homotopic to α by Proposition 2.2. Suppose that S includes another tight curve freely homotopic to α . Then it corresponds to a ϑ -invariant open line in Ω , which does not connect p and q . By Lemma 4.2(1), the open line should be a subset of $\text{bd}\Omega$. Since (1) gives the contradiction that α is freely homotopic to a component of δS , the uniqueness follows.

(3) Let p and q denote the attractor and the repeller of ϑ in Ω respectively. Suppose that no other fixed point of ϑ belongs to $\text{Cl}(\Omega)$. Similarly to (2), Lemma 4.2(4) and Proposition 2.2 imply that S includes a unique imbedded tight curve, which is strong. Suppose that a fixed point other than p or q belongs to $\text{Cl}(\Omega)$. Let it be denoted by r . It follows that ϑ -invariant segments in $\text{Cl}(\Omega)$ are \overline{pq} , \overline{pr} , and \overline{qr} . By Lemma 4.2(1), \overline{pr} and \overline{qr} are subsets of $\text{bd}\Omega$. Since Ω is not a triangle, we have $\overline{pq}^o \subset \Omega^o$. This line corresponds to an imbedded strong tight curve freely homotopic to α by Proposition 2.2. Exactly one of \overline{pr}^o and \overline{qr}^o is a subset of Ω by Lemma 4.2(5); it corresponds to a component of δS , which is a weak tight curve and is freely homotopic to α . Since \overline{pq}^o , \overline{pr}^o , and \overline{qr}^o are the only ϑ -invariant open lines in Ω , there is no other imbedded tight curve freely homotopic to α .

(4) It follows similarly to (3) that S includes a unique imbedded tight curve freely homotopic to α . By Lemma 4.2(2), the tight curve is a component of δS .

(5) This follows from (4).

(6) This follows from (2), (3), and (4).

(7) This follows from (6).

(8) Suppose that one of the components of δS is not a strong tight curve. Then S^o includes a unique imbedded strong tight curve β freely homotopic to the component by (6). Let E be the compact annulus

bounded by β and the component of δS . By Lemma 2.5 and (6), E is a trivial annulus or a boundary elementary annulus. Hence, S is not purely convex.

The converse result follows from (6).

(9) For each component of δS , there is a unique imbedded strong tight curve freely homotopic to it. Hence, either each component of δS is strong or S is the sum of S_0 and A_1, \dots, A_n , where S_0 is a surface such that components of δS_0 are strong tight curves, and A_i for each i is a compact annulus with convex boundary. In the former case, S is purely convex by (8), and we are done. Let us assume that we have the second case. Since S includes no imbedded π -annulus by Theorem 3.4, S_0 includes no imbedded π -annulus. Hence, S_0 is a convex surface by Theorem 3.4. By (8), S_0 is purely convex. Lemma 2.5 and (6) imply that each A_i is a boundary elementary annulus or a trivial annulus. The conclusions of (9) follow easily.

4.6. Let us end this section with the following corollary to Proposition 4.5. Let us assume now that S is not necessarily convex. Suppose that A , T , and P respectively are a compact annulus with geodesic boundary, a trivial annulus, and a purely convex surface in S . Since T and P are convex, every closed curve in T and P is a geodesic if and only if it is a tight curve (see §2). Notice also that if A is hyperbolic or quasi-hyperbolic, then the components of δA are imbedded tight curves by the annulus decomposition theorem in Appendix B.

Corollary. (1) *Each component of δP or δA is not a subset of T^o .*

(2) *Each component of δA or δT is not a subset of P^o .*

(3) *Each component of δT or δP is not a subset of A^o .*

Proof. (1) Suppose that a component of δA is a subset of T^o . Then A is hyperbolic or quasi-hyperbolic. Since components of δA are tight curves, Lemma 2.5(2) imply contradiction. Similarly, no component of δP is a subset of T^o .

(2) Let α and β be components of δA . Suppose that $\alpha \subset P^o$. Then A is hyperbolic or quasi-hyperbolic. We now deduce properties of $A \cap P$, which is nonempty. By Proposition 2.2, the set of boundary points of $A \cap P$ with respect to the relative topology of A is the union of tight curves that are components of δP or is the empty set. Hence, a component F of $A \cap P$ including α can be one of the following:

(a) a tight curve that is a common component of δA and δP ;

(b) a compact annulus B bounded by tight curves in A where one component of δB is a component of δP , and the other component of δB is a component of δA ; or

(c) A itself.

Since $\alpha \subset P^\circ$, (a) is not possible. Since Proposition 4.5(7) implies that α is not freely homotopic in P to a component of δP , (b) is not possible. Proposition 4.5(7) also implies that A is not a subset of P . Thus, (c) is also not possible. Therefore, no component of δA is a subset of P° .

A similar argument shows that a component of δT is not a subset of P° .

(3) Suppose that γ is a component of δT in A° . Thus, A is hyperbolic or quasi-hyperbolic, and $T^\circ \cap A^\circ \neq \emptyset$. Then (1) and Proposition 2.2 imply that the components α and β of δA are disjoint from T° . Hence $T^\circ \cap A^\circ = T^\circ \cap A$. Since $T^\circ \cap A$ is thus a relatively open and closed subset of T° , we have $T^\circ \cap A = T^\circ$ and $T^\circ \subset A$. Since T cannot be a subset of A , this is a contradiction.

Suppose that γ is a component of δP in A° . Thus, $P^\circ \cap A^\circ \neq \emptyset$. Then (2) and Proposition 2.2 imply similarly to the above paragraph that α and β are disjoint from P° . Similarly to the above, a contradiction follows. Thus the desired conclusion follows.

Let us remark about a consequence of the above corollary: given arbitrary two surfaces S_1 and S_2 among A , T , and P , if components of δS_1 and δS_2 intersect trivially in homotopy with one another, and A is hyperbolic or quasi-hyperbolic, then $S_1 \cap S_2$ is either empty or the union of common components of δS_1 and δS_2 .

5. The proof of the admissible decomposition theorem

In this section, we prove the admissible decomposition theorem. Let Σ be a compact real projective surface with convex boundary and negative Euler characteristic.

5.1. Let us denote the collection of trivial annuli in Σ by $T(\Sigma)$, and let $\mathcal{F}(\Sigma)$ denote the union of its elements. Recall from §3.10 that $M(\Sigma)$ denotes the collection of maximal annuli freely homotopic to π -annuli or hyperbolic or quasi-hyperbolic components of $\delta\Sigma$, and let $\mathcal{M}(\Sigma)$ denote the union of its elements. Let us denote by $C(\Sigma)$ the collection of the closures in Σ of components of $\Sigma - \mathcal{M}(\Sigma) - \mathcal{F}(\Sigma)$.

We claim the existence of an admissible decomposition collection: Σ is the sum of all subsurfaces that are elements of $T(\Sigma)$, $M(\Sigma)$, or $C(\Sigma)$. The elements of $C(\Sigma)$ are maximal purely convex surfaces.

Let us begin the proof. Consider two arbitrary elements of $T(\Sigma)$ or $M(\Sigma)$. Then they either are disjoint or intersect precisely at the union of their common boundary components by §§2.6 and 4.6 and Lemma 3.10.

Hence, $\mathcal{F}(\Sigma) \cup \mathcal{M}(\Sigma)$ is the union of compact annuli disjoint from one another. The topological boundary of the set is the union of imbedded tight curves in Σ° disjoint from one another. Thus, each element of $C(\Sigma)$ is a compact surface with convex boundary.

Let S be an element of $C(\Sigma)$. Suppose that S is an annulus. Since $\chi(\Sigma) < 0$, a component of δA for a maximal annulus A , an element of $M(\Sigma)$, equals a component of δS . The annulus decomposition theorem of Appendix B implies that S is the sum of elementary annuli and trivial annuli in S . Since a trivial annulus in S is a trivial annulus in Σ , it follows that S is the sum of elementary annuli. But this gives a contradiction that A is not maximal. Hence, $\chi(S) < 0$. Since every imbedded π -annulus in Σ is a subset of an element of $M(\Sigma)$, the surface S is convex by Theorem 3.4.

By construction, each component of δS either is a component of $\delta \Sigma$ or is a boundary tight-curve component of a maximal annulus in $M(\Sigma)$ or a trivial annulus in $T(\Sigma)$ in Σ° . Let us state two consequences of this:

- (1) Each component of δS is a strong tight curve.
- (2) Corollary 4.6 implies that S is not a proper subset of a purely convex surface.

Let us prove (1): Let α be an arbitrary component of δS .

(i) Suppose that α is a component of $\delta \Sigma$. If a trivial annulus or a boundary elementary annulus A in S includes α , then A is a trivial annulus in Σ or a boundary elementary annulus in Σ . A boundary elementary annulus in Σ is a subset of an element of $M(\Sigma)$ by §3.10. Since α is in the complement of $\mathcal{F}(\Sigma) \cup \mathcal{M}(\Sigma)$, it follows that α is not a subset of a trivial annulus or a boundary elementary annulus in S . By Proposition 4.5(9), α is a strong tight curve.

(ii) Suppose that α is a component of δA for a maximal annulus A in $M(\Sigma)$. If α is not strong, then α is a subset of a boundary elementary annulus in S by Proposition 4.5(9). This yields a contradiction that A is not maximal. Thus, α is a strong tight curve.

(iii) If α is a boundary tight-curve component of a trivial annulus in $T(\Sigma)$, then α is strong (see §3.1).

Consequently (1) and Proposition 4.5(8) imply that S is purely convex. By (2), S is a maximal purely convex surface. Therefore, the existence of an admissible decomposition collection follows.

5.2. Given two subsurfaces of a real projective surface, we say that they are *adjacent* if their intersection is the union of finitely many imbedded closed geodesics disjoint from one another.

We now claim the uniqueness: Suppose that T' , M' , and C' respectively are a collection of trivial annuli, a collection of maximal annuli, and a collection of maximal purely convex surfaces in Σ . If Σ is the sum of all the subsurfaces that are elements of T' , M' or C' , then $T' = T(\Sigma)$, $M' = M(\Sigma)$, and $C' = C(\Sigma)$.

By §2.6, $T' \subset T(\Sigma)$. Because of the decomposition of Σ into elements of T' , M' , and C' , the number of components of $\delta\Sigma$ that are not geodesic equals the number of elements of T' . Similarly, the number of nongeodesic components of $\delta\Sigma$ equals the number of the elements of $T(\Sigma)$. Thus, $T' = T(\Sigma)$.

Let A be an element of M' . Suppose that A is freely homotopic to a component of $\delta\Sigma$. Since $\chi(\Sigma) < 0$, Lemma 5.3 implies that at least one component of δA is a boundary component of a maximal purely convex surface in C' . Hence, A is hyperbolic or quasi-hyperbolic, and $A \in M(\Sigma)$. Suppose now that A is not freely homotopic to a component of $\delta\Sigma$. Then each component of δA is a boundary component of a maximal purely convex surface in C' . Hence, each component of δA is strong by Proposition 4.5(8), and A includes a π -annulus by Lemma 5.5. Thus, $A \in M(\Sigma)$, and $M' \subset M(\Sigma)$.

Suppose that M' is a proper subset of $M(\Sigma)$. Then by Lemma 3.10, an element A of $M(\Sigma) - M'$ is a subset of the union of some purely convex surfaces in C' and trivial annuli. Since the intersection of A with each trivial annulus is a geodesic boundary component of the trivial annulus or is empty by §4.6, A is a subset of the union of purely convex surfaces belonging to C' . Since by Lemma 5.3, no two elements of C' are adjacent, A is a subset of a purely convex surface belonging to C' . This contradicts Proposition 4.5(7). Therefore, $M' = M(\Sigma)$. Since no two elements of $C(\Sigma)$ are adjacent, we also have $C' = C(\Sigma)$. Hence, the uniqueness is proved, and the proof of the admissible decomposition theorem is complete.

5.3. Lemma. *Suppose that two admissible subsurfaces in Σ are adjacent. Then their types are different from each other.*

Proof. It is straightforward to prove this with the help of the following lemma.

5.4. The following lemma generalizes Theorem 3.7 of Goldman [16]. It is also true that the lemma may be proved by an extension of the proof given by Goldman [16].

Lemma. *Let S_1, \dots, S_n be convex compact subsurfaces of Σ . Suppose that $\chi(S_i) \leq 0$ for each i , $i = 1, \dots, n$, and that $n \geq 2$. Assume the following conditions:*

- $\{S_1, \dots, S_n\}$ is a decomposition collection of a connected subsurface S' of Σ .
- An annulus in the decomposition collection is not adjacent to another annulus in the decomposition collection.
- $S_i \cap S_j$ whenever $i \neq j$ is either the union of strong tight curves in S or empty.

Then S' is a convex subsurface of Σ . Moreover, if S_1, \dots, S_n are purely convex, then S' is purely convex.

Proof. The union S' is a compact subsurface of S with convex boundary and with $\chi(S') < 0$. Suppose that S' is not convex. Then S' includes an imbedded π -annulus. Let us call it E . There exists an integer k , $1 \leq k \leq n$, such that $E^\circ \cap S_k \neq \emptyset$. By Lemma 3.5, E is not a subset of S_k . Therefore, $\delta S_k \cap E^\circ \neq \emptyset$. Let α be an arbitrary component of δS_k . Each component of δE either is disjoint from α or is identical with α by Lemma 3.8. This fact implies that E° includes a component β of δS_k . The second condition of the premise shows that β is a component of δS_l for some l where $\chi(S_l) < 0$. Since β is a strong tight curve, it follows that β is a boundary component of a purely convex surface in S_l by Proposition 4.5(9). However, this contradicts Corollary 4.6. Hence, S' is convex, and the desired conclusions follow.

5.5. Lemma. *Let A be a compact hyperbolic or quasi-hyperbolic annulus such that each component of δA is a strong tight curve. Then A includes a π -annulus.*

Proof. By the annulus decomposition theorem of Appendix B, A is the sum of elementary annuli E_i , $i = 1, \dots, n$, where $E_i \cap E_j = \emptyset$ if $|i - j| > 1$, and $E_i \cap E_{i+1}$ is a common component of δE_i and δE_{i+1} . Suppose that $n = 1$. Since the boundary components are strong, E_1 is an elementary annulus of type IIa, a π -annulus. Assume now that $n \geq 2$. Suppose that A is hyperbolic. Then E_1 and E_2 are elementary annuli of type I and meet at a weak tight curve. This implies that $E_1 \cup E_2$ is a π -annulus. Suppose that A is quasi-hyperbolic. If E_1 is an elementary annulus of type IIa, then E_1 is a π -annulus. If E_1 is an elementary annulus of type IIb, then so is E_2 , and E_1 and E_2 meet at a weak tight curve. It follows that $E_1 \cup E_2$ is a π -annulus. In all cases, A includes a π -annulus.

Appendix A: The topology of closed curves in surfaces

1. Let S be a surface. Let \tilde{S} be the universal cover of S , and let $\text{pr}: \tilde{S} \rightarrow S$ denote the universal covering map. Let S^1 be a circle, and

let $c: \mathbf{R} \rightarrow \mathbf{S}^1$ denote the infinite cyclic covering map from the real line \mathbf{R} to \mathbf{S}^1 . A *closed curve* in S is a regular arc from \mathbf{S}^1 to S . It is said to be *simple* if it is injective. (An *imbedded closed curve* is the image submanifold of an injective closed curve.) In this paper, we consider only closed curves that map into S^o or δS . (Closed geodesics in real projective surfaces always have this property; see §2.1 of the main text.) A *lift* of a closed curve $\alpha: \mathbf{S}^1 \rightarrow S$ to \tilde{S} is a map $\tilde{\alpha}$ satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\tilde{\alpha}} & \tilde{S} \\ c \downarrow & & \text{pr} \downarrow \\ \mathbf{S}^1 & \xrightarrow{\alpha} & S. \end{array}$$

A *lift* of α to a cover S' of S is the map α' satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathbf{S}^1 & \xrightarrow{\alpha'} & S' \\ \parallel & & \text{pr}' \downarrow \\ \mathbf{S}^1 & \xrightarrow{\alpha} & S, \end{array}$$

where $\text{pr}': S' \rightarrow S$ is the covering map. (It will be clear from the context which one we mean by a "lift.")

A closed curve is said to be *essential* if it is not null-homotopic. Two closed curves α and β are said to *intersect* if their images intersect. α *intersects* a set A if the image of α intersects A . α and β are said to *intersect trivially in homotopy* if they are homotopic to disjoint closed curves and are said to *intersect essentially* otherwise. Simple closed curves α and β have *minimal intersection* if their image submanifolds α_1 and β_1 have minimal intersection (see Casson and Bleiler [3]); i.e., α_1 and β_1 intersect transversally and there are no arcs μ and ν in α_1 and β_1 respectively having common endpoints and such that $\mu \cup \nu$ is the boundary of a disk in S .

Two closed curves $\alpha: \mathbf{S}^1 \rightarrow S$ and $\beta: \mathbf{S}^1 \rightarrow S$ are said to be *freely homotopic* if α and β are homotopic. We need the following definitions for convenience in the main text. A closed curve $\alpha: \mathbf{S}^1 \rightarrow S$ is said to be *freely homotopic* to an imbedded closed curve β in S if α is homotopic to an imbedding from \mathbf{S}^1 realizing β . Two imbedded closed curves α and β in S are said to be *freely homotopic* if an imbedding from \mathbf{S}^1 realizing α is homotopic to that realizing β . An annulus A in S is *freely homotopic* to α if a component of δA is freely homotopic to α . Two

annuli A and B in S are *freely homotopic* to each other if a component of δA is freely homotopic to a component of δB .

2. We now prove needed facts on curves in surfaces using hyperbolic structures. Let \mathbf{H}^2 be the hyperbolic plane. We identify \mathbf{H}^2 with the interior of the upper hemisphere of the complex projective space \mathbf{CP}^1 using the Poincaré model. A hyperbolic structure on S with convex boundary whose associated metric is complete and such that each essential closed curve has hyperbolic holonomy is called a *generic hyperbolic structure* on S . Suppose that S is homeomorphic to a compact annulus or a cover of a compact surface of negative Euler characteristic. Since a compact surface with negative Euler characteristic admits a generic hyperbolic structure (see [27] and [3]), it follows that S admits a generic hyperbolic structure. Further, given an essential simple closed curve in S , there is a generic hyperbolic structure on S such that the closed curve is geodesic (see Casson and Bleiler [3]). We may identify \tilde{S} with a convex domain Ω in \mathbf{H}^2 complete under the hyperbolic metric. Then S is identified with Ω/Γ where Γ is a discrete subgroup of the group $\text{PSL}(2, \mathbf{R})$ of fractional linear transformations with real coefficients acting properly discontinuously and freely on Ω . Here, Γ is identified with the deck transformation group of \tilde{S} . (Each element of Γ is hyperbolic.)

Let us first discuss about commuting elements of Γ . Let ϑ and φ be two nontrivial elements of Γ . Then ϑ and φ commute if and only if ϑ and φ have the same fixed point set (see Proposition I.D.3 in [21]). This means that ϑ and φ commute if and only if ϑ and φ have a common invariant complete geodesic in \mathbf{H}^2 .

We claim that ϑ commutes with φ if and only if ϑ and φ are multiples of a common element of Γ . Suppose that ϑ and φ commute. \mathbf{H}^2 includes a complete geodesic l that is invariant under both ϑ and φ . Since the action of the group $\{\varphi' \in \Gamma \mid \varphi'(l) = l\}$ on l is properly discontinuous and free, the group is an infinite cyclic group. Thus, ϑ and φ are multiples of its generator. The converse portion of the statement is trivial to show.

Lastly, suppose that ϑ commutes with $\varphi \circ \vartheta \circ \varphi^{-1}$. Then we claim that ϑ commutes with φ . The proof is as follows: ϑ has an invariant complete geodesic l in \mathbf{H}^2 . Since $\varphi \circ \vartheta \circ \varphi^{-1}$ commutes with ϑ , it also leaves l invariant. Since l is the unique invariant complete geodesic of ϑ , we have $\varphi(l) = l$. Hence, φ commutes with ϑ .

3. We now discuss on lifts of closed curves. Let r and s be the endpoints of a complete geodesic l corresponding to an essential simple

closed curve. Then given ϑ in Γ , we have either that $\vartheta(r) = r$ and $\vartheta(s) = s$ or that $\vartheta(r)$ and $\vartheta(s)$ lie in a common component of $\text{bd } \mathbf{H}^2 - \{r, s\}$.

Lemma. *Suppose that α is an essential closed curve freely homotopic to a finite covering of a simple closed curve, and that $\tilde{\alpha}$ is a simple arc that is the image of a lift of α to \tilde{S} , an injective map. Suppose that $\tilde{\alpha}$ and $\vartheta(\tilde{\alpha})$ for a deck transformation ϑ are transversal. Then if $\tilde{\alpha}$ and $\vartheta(\tilde{\alpha})$ intersect at a point, then they intersect at another point.*

Proof. Since α is essential, $\tilde{\alpha}$ is an arc ending at two distinct points r and s of $\text{bd } \mathbf{H}^2$ (see the proof of Lemma 2.3 of Casson and Bleiler [3]). Let φ be an element of Γ corresponding to α and $\tilde{\alpha}$. Then r and s are the endpoints of a φ -invariant complete geodesic. Suppose that we have $r = \vartheta(r)$ and $s = \vartheta(s)$. Then ϑ and φ commute. We have thus $\vartheta^m = \varphi^n$ for two nonzero integers m and n by §2. This means that $\langle \varphi^n \rangle$ acts on both $\tilde{\alpha}$ and $\vartheta(\tilde{\alpha})$. Hence, the conclusion follows. Suppose now that $\vartheta(r)$ and $\vartheta(s)$ lie in the same component of $\text{bd } \mathbf{H}^2 - \{r, s\}$. The conclusion follows easily in this case by transversality.

4. We will need the following lemma. Let d denote the complete hyperbolic metric on S corresponding to the generic hyperbolic structure. We denote the induced metric on an arbitrary cover of S by d also.

Lemma. *Suppose that S is compact. Let α be an essential simple closed curve in S , and let $\tilde{\alpha}$ be the simple arc that is the image of a lift of α to \tilde{S} , an injective map. Let ϑ be a deck transformation of \tilde{S} corresponding to α and $\tilde{\alpha}$. Suppose that a component \tilde{S}_α of $\tilde{S} - \tilde{\alpha}$ is homeomorphic to an open disk. Then there is a deck transformation φ not commuting with ϑ , that maps a point of $\tilde{\alpha}$ to a point of \tilde{S}_α .*

Proof. We may assume without loss of generality that α is geodesic. Since \tilde{S}_α is an open disk, the completeness of \tilde{S} implies that \tilde{S}_α is a component of \mathbf{H}^2 removed the complete geodesic $\tilde{\alpha}$. Hence, there is a sequence of points $\{x_i\}$ in \tilde{S}_α such that $\{d(x_i, \tilde{\alpha})\} \rightarrow \infty$. Let $x \in \tilde{\alpha}$. Since S is compact, for each i , there is a deck transformation ϑ_i such that $d(\vartheta_i(x), x_i) < 2 \text{diam}(S)$ for the d -diameter $\text{diam}(S)$. Let i be an integer such that $d(x_i, \tilde{\alpha}) > 2 \text{diam}(S)$. Then $\vartheta_i(x) \in \tilde{S}_\alpha$. This completes the proof (compare to [1]).

5. Let S' be a covering of S with covering map $p: S' \rightarrow S$. (S is not necessarily compact.)

Lemma. *Suppose that α and β are essential imbedded closed curves freely homotopic to each other in S' . Suppose that $\alpha \neq \beta$ and that $p(\alpha)$ and $p(\beta)$ are imbedded closed curves. Then $p(\alpha)$ and $p(\beta)$ are not identical.*

Proof. We may assume without loss of generality that $p(\alpha)$ is a closed geodesic. Suppose that $p(\alpha) = p(\beta)$. The covering map p induces a generic hyperbolic structure on S' , where α and β are imbedded closed geodesics in S' distinct from each other. However, it is well known that an essential simple closed curve in a hyperbolic surface with convex boundary is freely homotopic to at most one imbedded closed geodesic (see Lemma 2.3 in [3]). This is a contradiction.

6. Given a topologically imbedded submanifold S_1 of S , let S'_1 be a component of $p^{-1}(S_1)$. Then $p|S'_1$ is a covering map onto S_1 .

Suppose that α and β are essential simple closed curves in S' disjoint from and freely homotopic to each other. Suppose that $p|\alpha$ and $p|\beta$ are imbeddings onto $p(\alpha)$ and $p(\beta)$ respectively, and that $p(\alpha)$ and $p(\beta)$ are disjoint.

Now, $p(\alpha)$ and $p(\beta)$ bound a unique annulus, say A , and α and β also bound a unique annulus, say A' . Each component of $p^{-1}(A)$ is a cover of A and, hence, is homeomorphic to either $S^1 \times I$ or $\mathbf{R} \times I$ for a compact interval I . Let F be the component containing α . Then α is a component of δF . It follows that F is homeomorphic to $S^1 \times I$, and the covering map $p|F$ is an imbedding onto A . Let β' be the other component of δF . Then $p(\beta') = p(\beta)$. By the preceding Lemma 5, $\beta' = \beta$. Since α and β bound a unique compact annulus, $F = A'$. It follows that $p|A'$ is an imbedding onto A .

Appendix B: The annulus decomposition theorem

1. In this section, we will prove the annulus decomposition theorem (Corollary 3.6 of [14]). The version proved here is more general than that of Goldman since we do not require the boundary components to be principal geodesic and the holonomy to be hyperbolic. We will base our argument on the articles of Goldman [13], [14] (see [9] also) except for Proposition 2, which is not a step used by Goldman. (The author benefited greatly from a conversation with W. Goldman in February 1993 in constructing the proof.) For examples of projective annuli, refer to Sullivan and Thurston [26] and §3 of Goldman [14].

Note that we will need only the materials in §1, the introduction of §2, and §2.1 in the main text. The results in this appendix are independent of other parts of the main text and Appendix A.

Annulus decomposition theorem. *Let A be a hyperbolic or quasi-hyperbolic projective annulus with convex boundary. Then A decomposes into*

subsurfaces that are elementary annuli or trivial annuli.

Let $p: \tilde{A} \rightarrow A$ be the universal covering map. Let A have a complete metric d . We may assume that d is a hyperbolic metric with respect to which δA is convex. We denote also by d the induced complete metric on \tilde{A} . Let $\mathbf{dev}: \tilde{A} \rightarrow \mathbf{S}^2$ be a developing map, and let ϑ be the projective automorphism corresponding to a generator φ of the deck transformation group of \tilde{A} . Let μ be the spherical Riemannian metric on \tilde{A} induced from \mathbf{S}^2 by \mathbf{dev} , and let \mathbf{d} be the induced metric on \tilde{A} . Let \tilde{A} be the completion. (See [18] and the introduction of [5].)

Suppose that ϑ is hyperbolic (leaving aside the case where ϑ is quasi-hyperbolic until the end). Let $s, m, w, -s, -m,$ and $-w$ denote the fixed points of ϑ as in §1.4 of the main text. There are three invariant great circles $l_1, l_2,$ and l_3 . We use the decompositions used in Goldman's article [14, §2.1]. (See [12] also.) For each i , since l_i is ϑ -invariant, $\mathbf{dev}^{-1}(l_i)$ is φ -invariant. Let $L_i = p(\mathbf{dev}^{-1}(l_i))$. Then $p^{-1}(L_i) = \mathbf{dev}^{-1}(l_i)$, L_i is a compact set, and each component of L_i is a one-dimensional compact submanifold of A or the set of a point of δA . Each one-dimensional manifold component is either an imbedded closed geodesic or a maximal segment. Either the component is a subset of δA , or its manifold interior is a subset of A° by Lemma 3.4 in [5]. Note that $\mathbf{dev}^{-1}(l)$ is also φ -invariant. Let $L = p(\mathbf{dev}^{-1}(l))$. Then L is the union of one-dimensional submanifolds transversal to one another and the finite set of points of δA . Notice also that $p^{-1}(L) = \mathbf{dev}^{-1}(l)$.

2. Our first step is as follows:

Proposition. *There exists a component of L that is an imbedded tight curve freely homotopic to a component of δA .*

Proof. Suppose that A is convex. Then $\mathbf{dev}: \tilde{A} \rightarrow \mathbf{S}^2$ is an imbedding onto a convex domain Ω by Lemma 1.5 of [5]. A is projectively homeomorphic to $\Omega/\langle\vartheta\rangle$. Since Ω is ϑ -invariant, an attractor and a repeller of ϑ are in $\text{bd}\Omega$. We may assume without loss of generality that they are s and w respectively. Suppose that $\overline{sw}^o \not\subset \Omega$. Then $\overline{sw} \subset \text{bd}\Omega$ and $\overline{sw}^o \cap (\text{bd}\Omega - \Omega) \neq \emptyset$, which imply that $\Omega/\langle\vartheta\rangle$ is not compact. Hence, \overline{sw}^o is a subset of Ω and corresponds to an imbedded tight curve. Clearly, it is a component of L .

Suppose that A is not convex. Then as in §5.1 of [5], \tilde{A} includes a triangle T such that $T \cap \tilde{A}_\infty \subset \eta^o \cap \tilde{A}_\infty \neq \emptyset$ holds for an edge η of T . Let α be a maximal line in $\eta - \tilde{A}_\infty$ containing an endpoint of η . Let α be oriented away from the endpoint of η . Since $p|\alpha$ is semi-infinite in d , there is a point x in A and a d -unit vector \mathbf{v} at x such that

$p|\alpha$ passes arbitrarily close to x and such that its d -unit direction vector passes arbitrarily near \mathbf{v} (see Fried [11]). This implies that there is a sequence $\{\alpha_i\}$ of lines in \tilde{A} such that the following properties hold:

(1) For each i , there is an integer $n(i)$ such that $\alpha_i = \varphi^{n(i)}(\alpha)$. We have $\{|n(i)|\} \rightarrow \infty$ as $i \rightarrow \infty$.

(2) There is a sequence $\{x_i\}$, $x_i \in \alpha_i$ for each i , converging to a point \tilde{x} of \tilde{A} corresponding to x .

(3) The sequence $\{\mathbf{v}_i\}$, where \mathbf{v}_i is the d -unit direction vector of α_i at x_i for each i , converges to a vector $\tilde{\mathbf{v}}$ at \tilde{x} corresponding to \mathbf{v} .

Let α_∞ be the maximal line in \tilde{A} passing through \tilde{x} in the direction of $\tilde{\mathbf{v}}$. We will prove that α_∞ corresponds to an imbedded tight curve in L that we seek.

First, let A' be a real projective open annulus including A . Clearly, \tilde{A} is a closed proper subset of \tilde{A}' , an open disk. The metric d on \tilde{A} extends to a complete hyperbolic metric on \tilde{A}' such that the inclusion map from \tilde{A} to \tilde{A}' is isometric. Let us denote the complete metric on \tilde{A}' by d also. The developing map dev extends to a developing map on \tilde{A}' , which is denoted by dev also. Thus, the spherical metric μ also extends to a spherical metric on \tilde{A}' induced by dev . Let us denote the Riemannian metric by μ also.

Considering \tilde{A}' as a Riemannian manifold with the spherical metric μ , we obtain a differentiable exponential map for projective geodesics defined on an open domain in $T(\tilde{A}')$ and mapping to \tilde{A}' . We may reparametrize every projective geodesic so that it is d -length parametrized. It is easy to see that there is an associated continuous map $\text{exp}' : T(\tilde{A}') \rightarrow \tilde{A}'$ where $\text{exp}'\{t\mathbf{u}\mid t \in \mathbf{R}\}$ is a d -length parametrized projective geodesic for each d -unit vector \mathbf{u} in $T(\tilde{A}')$.

Next, we claim that α_∞ is infinitely long with respect to d . Suppose not. Then α_∞ is a segment of finite d -length. Let y be the endpoint of α_∞ in the direction of $\tilde{\mathbf{v}}$. Let \tilde{l} be the d -length of the segment on α_∞ with endpoints \tilde{x} and y . Let us choose a point y_i in the infinite component of $\alpha_i - \{x_i\}$ for each i such that the d -length of the segment on α_i with endpoints x_i and y_i is \tilde{l} . The above paragraph implies that $\{y_i\} \rightarrow y$ in \tilde{A}' . Since α_i is semi-infinite, α_i includes a segment ζ_i with $\zeta_i^o \ni y_i$ such that the d -distance along α_i from y_i to each point of $\delta\zeta_i$ is equal to \tilde{l} . Again, the above paragraph shows that $\{\zeta_i\}$ converges to a segment ζ in \tilde{A}' such that $\zeta^o \ni y$, and ζ given an orientation is in the same direction as α_∞ at y . Since $\zeta_i \subset \tilde{A}$ for each i , it follows

that $\zeta \subset \tilde{A}$, which gives a contradiction that α_∞ is not maximal in \tilde{A} . Therefore, α_∞ is infinite with respect to d .

For some k , $k = 1, 2, 3$, l_k includes $\mathbf{dev}(\alpha_\infty)$. Thus, $p|_{\alpha_\infty}$ maps into L_k . Since $p|_{\alpha_\infty}$ is infinite in d , it maps onto an imbedded closed geodesic that is a component of L_k . Since α_∞ is a maximal line, it follows that $p|_{\alpha_\infty}$ is a covering map onto the imbedded closed geodesic.

Finally, we claim that α_∞ is a convex line. Suppose not. Then the \mathbf{d} -length of α_∞ is greater than π . In fact, since $p|_{\alpha_\infty}$ covers an imbedded closed geodesic, a closed one-dimensional real projective manifold, it is straightforward to show that α_∞ is infinitely long in \mathbf{d} in both directions. Each α_i is a convex line and a geodesic of \mathbf{d} . Hence, α_∞ includes a convex segment β such that $\mathbf{dev}(\beta)$ is the geometric limit of a subsequence of $\{\mathbf{Cl}(\mathbf{dev}(\alpha_i))\}$ and such that $\beta \ni \tilde{x}$. Since β is convex, $\mathbf{dev}|_\beta$ is injective; thus, \tilde{A} includes a d -bounded open neighborhood \mathcal{U} of β such that $\mathbf{dev}|_{\mathcal{U}}$ is an imbedding. For infinitely many i , it is easy to see that $\mathbf{dev}(\alpha_i) \subset \mathbf{dev}(\mathcal{U})$ and, hence, $\alpha_i \subset \mathcal{U}$. Since this contradicts the fact that α_i for each i is infinite with respect to d , it follows that α_∞ is convex, and, therefore, $p|_{\alpha_\infty}$ is a covering map onto the imbedded tight curve. This implies the conclusion of Proposition 2.

3. To continue, we need several topological facts. The first collection of facts are as follows: By a *simple piecewise-regular curve*, we mean a topologically imbedded curve that is the union of finitely many regular curves transversal to one another. Let S be a compact real projective surface with convex boundary whose components are simple piecewise-regular curves. Let F be a compact subset consisting of components ζ_1, \dots, ζ_n satisfying the following conditions:

- (1) Each ζ_i is an imbedded geodesic curve unless it is the set of a point of δS .
- (2) Each $\zeta_i \cap \delta S$ is either the set of endpoints of ζ_i or ζ_i itself or the empty set.

Suppose that S includes no imbedded closed curve intersecting F at a point transversally and intersecting F at no other point. Then it is straightforward to show that given a component R of $S - F$, the closure $\mathbf{Cl}(R)$ is a compact surface with convex boundary whose components are simple piecewise-regular curves and that $\mathbf{Cl}(R)^\circ = R^\circ$.

The second collection of facts that we need are as follows: Suppose that B and its closure B_1 are topologically imbedded subsurfaces of S such that $B_1^\circ = B$. Let S' be a covering of S with covering map $q: S' \rightarrow S$. Let \mathcal{B} be a component of $q^{-1}(B)$. Then $q|_{\mathcal{B}}$ is a covering map onto

B . Let \mathcal{B}_1 be a component of $q^{-1}(B_1)$ including \mathcal{B} . Also, $q|_{\mathcal{B}_1}$ is a covering map onto B_1 . Clearly, $\mathcal{B} \subset \mathcal{B}_1^o$. Since $q(\mathcal{B}_1^o) \subset B$, we have $\mathcal{B}_1^o \subset q^{-1}(B)$. Since \mathcal{B}_1^o is path-connected, we conclude $\mathcal{B} = \mathcal{B}_1^o$ and $\text{Cl}(\mathcal{B}) = \mathcal{B}_1$.

The third set of facts are as follows: Let K be a closed subset of S , and let H be a component of $S^o - K$. Let $K' = q^{-1}(K)$, and let H' be a component of $q^{-1}(H)$. Then H' is a subset of $S^{oo} - K'$. Let H'' be a component of $S^{oo} - K'$ including H' . Since $q(H'')$ is path-connected and disjoint from K , we have $q(H'') \subset H$. Thus, $H'' \subset q^{-1}(H)$. Since H'' is path-connected, $H' = H''$. We conclude that H' is a component of $S^{oo} - K'$.

Final facts are as follows: Let M be a surface, K a closed subset of M , and H a component of $M^o - K$. Then $(\text{Cl}(H) \cap M^o) - H \subset K$. The proof is as follows: Let x be a point of a component H' of $M^o - K$ different from H . Then H' is an open subset of $M^o - K$ and is an open subset of M^o . Since H and H' are disjoint, x does not belong to $\text{Cl}(H)$. Hence, if a point x belongs to $\text{Cl}(H) \cap M^o$, then x belongs to H or K . Thus the claim follows.

4. Let B be a component of $A - L$. We will prove properties of B_1 where $B_1 = \text{Cl}(B)$. (Recall the facts on L in §1 of this appendix.)

Proposition. *Suppose that B_1 includes an imbedded tight curve. Then B_1 is a trivial annulus or an elementary annulus, where $B_1 \cap L$ equals the union of all geodesic components of δB_1 .*

Proof. We claim that B_1 is a compact surface such that $B^o = B_1^o$. To prove this, we apply the results of §3 to B_1 . First consider $A - L_1$. Choose a component R_1 including B . Note that R_1 is disjoint from L_1 . Then by the above argument, $\text{Cl}(R_1)$ is a compact surface with convex boundary whose components are simple piecewise-regular curves, and $\text{Cl}(R_1)^o = R_1^o$. Consider $\text{Cl}(R_1) - L_2$. We obtain a component R_2 including B . Similarly, $\text{Cl}(R_2)$ is a compact surface with convex boundary whose components are simple piecewise-regular curves, and $\text{Cl}(R_2)^o = R_2^o$. Since $R_2 \subset \text{Cl}(R_1) - L_2$, it follows that $R_2^o \subset \text{Cl}(R_1)^o - L_2 = R_1^o - L_2$. Hence, R_2^o is disjoint from L_1 and L_2 since R_2^o is a subset of R_1^o and R_2 . Finally, we obtain a component R_3 of $\text{Cl}(R_2) - L_3$ including B . Similarly, $\text{Cl}(R_3)$ is a compact surface with convex boundary, and $\text{Cl}(R_3)^o = R_3^o$. Since $R_3 \subset \text{Cl}(R_2) - L_3$, we have $R_3^o \subset R_2^o - L_3$. Hence, R_3^o is disjoint from L_1 , L_2 , and L_3 since R_3^o is a subset of R_2^o and R_3 . Clearly, $B^o \subset R_3^o$. Since B^o is a component of $S^o - L_1 - L_2 - L_3$, we have $B^o = R_3^o$. Hence, $B_1 = \text{Cl}(R_3)$ and $B_1^o = B^o$. Thus, our claim follows.

Next, let B' be a component of $p^{-1}(B^o)$ for the universal covering map $p: \tilde{A} \rightarrow A$, and let \tilde{B}_1 be the component of $p^{-1}(B_1)$ that includes B' . By Lemma 5, $\text{dev}|_{B'}$ is an imbedding onto a convex open domain \mathcal{O} in an open ϑ -invariant triangle T . Let $\text{Cl}(B')$ denote the closure of B' in \tilde{A} . Since B' is convex, §1.4 of [5] implies that $\text{dev}|\text{Cl}(B')$ is an imbedding onto a convex compact subset $\text{Cl}(\mathcal{O})$ of $\text{Cl}(T)$. Section 3 shows that $\tilde{B}_1 = \text{Cl}(B') \cap \tilde{A}$. Therefore, $\text{dev}|\tilde{B}_1$ is an imbedding onto a dense domain in $\text{Cl}(\mathcal{O})$, which is a subsurface denoted by Ω .

Finally, we use the hypothesis that B_1 includes an imbedded tight curve. B_1 is projectively homeomorphic to the quotient surface of Ω by the action of a discrete group of projective automorphisms. Since B_1 is not simply connected and includes an imbedded tight curve, the discrete group is not trivial and equals $\langle \vartheta \rangle$. We may assume without loss of generality that s and w are the attractor and the repeller of ϑ in $\text{Cl}(T)$ respectively. Since $\text{Cl}(\mathcal{O})$ is a ϑ -invariant convex compact subset of $\text{Cl}(T)$, it follows that $\text{Cl}(\mathcal{O})$ equals either $\text{Cl}(T)$ or a convex compact subset K whose boundary is the union of \overline{sw} and an open arc γ in T connecting s and w . In order that $\Omega/\langle \vartheta \rangle$ be compact, Ω must either equal $T \cup \overline{sw}^o \cup \eta^o$ for an edge η of T distinct from \overline{sw} , or equal $K - \{s, w\}$. Since the quotients of these sets are an elementary annulus or a primitive trivial annulus respectively, B_1 is an elementary annulus or a trivial annulus. Suppose that B_1 is a trivial annulus. The nongeodesic component of δB_1 is disjoint from L since $\gamma \subset T$. Clearly, the geodesic component of δB_1 lies in L , and B_1^o is disjoint from L . Hence, $B_1 \cap L$ is the geodesic component of δB_1 . If B_1 is an elementary annulus, then a similar argument shows that $B_1 \cap L = \delta B_1$. This completes the proof of Proposition 4.

5. Lemma. *The subset B' is a convex subsurface of \tilde{A} , and $\text{dev}|_{B'}$ is an imbedding onto a convex subset of an open ϑ -invariant triangle.*

Proof. Since the connected set B' lies in the complement of $\text{dev}^{-1}(l)$, it follows that $\text{dev}(B')$ is a subset of an open ϑ -invariant triangle, say T . For each point x of B' , we denote by E_x the subset of the points of B' reachable from x by segments in B' . Similarly to Proposition 1.2 of Carrière [2], it follows that $\text{dev}|_{E_x}$ is an imbedding onto an open domain in T for each point x of B' . We let E_x^* denote the image of E_x under dev . Similarly to Proposition 1.3.2 of Carrière or Lemma 3 of Koszul [19], B' is convex if and only if E_x^* is a convex subset of T for each point x of B' .

Suppose that B' is not convex. Then E_x^* is not convex for a point x of B' . Hence, T includes a compact triangle R such that

$$R \cap (T - E_x^*) \subset \eta^o \cap (T - E_x^*) \neq \emptyset$$

holds for an edge η of R (see [2]). The injectivity of $\text{dev}|_{E_x}$ implies that B' includes a convex open triangle imbedding onto R^o under dev . Taking the closure of the open triangle and using §1.4 of [5], we obtain a triangle R' in $\text{Cl}(B')$, a subset in \tilde{A} , such that

$$R' \cap (\tilde{A} - B') \subset \zeta^o \cap (\tilde{A} - B') \neq \emptyset$$

holds for an edge ζ of R' .

Let $L' = \text{dev}^{-1}(l)$. Since $p^{-1}(L) = L'$, and B^o is a component of $A^o - L$, from §3 it follows that B' is a component of $\tilde{A}^o - L'$. Thus,

$$(\text{Cl}(B') \cap \tilde{A}^o) - B' \subset L', \quad (\zeta^o \cap \tilde{A}^o) - B' \subset L'.$$

Since $\tilde{A} = \tilde{A}_\infty \cup \delta\tilde{A} \cup \tilde{A}^o$, we have

$$\zeta^o \cap (\tilde{A} - B') \subset \zeta^o \cap (\tilde{A}_\infty \cup \delta\tilde{A} \cup L').$$

Let α be the component of $\zeta \cap B'$ containing a vertex of ζ . Let y be the endpoint of α not in α ; that is, y is the unique point of $\text{Cl}(\alpha) - \alpha$ in \tilde{A} . By our choice, y is a point of $\zeta^o \cap (\tilde{A} - B')$. Then y is not an element of δA ; otherwise α is tangent to a component of δA , and $\alpha \subset \delta A$. Also, y is not an element of $\text{dev}^{-1}(l)$ since $R \subset T$. Hence, y is a point of \tilde{A}_∞ . Now, this implies that α is semi-infinite in d . Again $p|\alpha$ is semi-infinite in d . We deduce as in the proof of Proposition 2 that there is a sequence $\{\alpha_i\}$ of d -infinite lines in \tilde{A} having the following properties:

(1) For each i , there is an integer $n(i)$ such that $\alpha_i = \varphi^{n(i)}(\alpha)$. We have $\{|n(i)|\} \rightarrow \infty$ as $i \rightarrow \infty$.

(2) There is a sequence $\{x_i\}$, where $x_i \in \alpha_i$ for each i , converging to a point \tilde{x} of \tilde{A} .

About the sequence we will now derive two facts that contradict each other. The first is as follows: There is a tiny disk $B(\tilde{x})$ of \tilde{x} . Clearly, $\mathbf{d}(\tilde{x}, \text{bd} B(\tilde{x})) = \varepsilon$ for a positive constant ε . Let N be an integer such that $\mathbf{d}(x_i, \tilde{x}) < \varepsilon/2$ for $i > N$. Since α_i is semi-infinite in d , it follows that for each i greater than N , $\alpha_i \cap B(\tilde{x})$ includes a line whose \mathbf{d} -length is bounded below by $\varepsilon/2$. Thus, the \mathbf{d} -length of $\text{dev}(\alpha_i)$ for each i is bounded below by a positive constant independent of i .

Second, let us consider the image of α_i under dev . For each i , there is the integer $n(i)$ such that $\text{dev}(\alpha_i) = \vartheta^{n(i)}(\text{dev}(\alpha))$. We may assume without loss of generality that $\{n(i)\} \rightarrow \infty$ or that $\{n(i)\} \rightarrow -\infty$ by extracting subsequences, if necessary. Since both endpoints of $\text{dev}(\alpha)$ belong to T , the sequence $\{\text{Cl}(\text{dev}(\alpha_i))\}$ converges to the set of the attractor or the set

of the repeller of ϑ in $\text{Cl}(T)$. Hence, $\{\mathbf{d}\text{-length}(\text{dev}(\alpha_i))\}$ converges to 0.

Since the above two paragraphs contradict each other, we conclude that B' is convex. Since B' is convex, it follows by [5] that $\text{dev}|_{B'}$ is an imbedding onto a convex open domain in T .

6. We can now prove the annulus decomposition theorem. By Proposition 2, there is a component B of $A-L$ such that its closure B_1 includes an imbedded tight curve. By Proposition 4, B_1 is an elementary annulus or a trivial annulus. Each component of δB_1 is a subset of either A^o or δA . Let B' be any other component of $A-L$ sharing a boundary point with B . Since each component of δB in A^o is an imbedded tight curve that is a component of L , the closure of B' also includes an imbedded tight curve. By Proposition 4, $\text{Cl}(B')$ is also a trivial annulus or an elementary annulus. Thus, an induction shows that the closure of each component of $A-L$ is a trivial annulus or an elementary annulus. This implies the conclusion of the annulus decomposition theorem if ϑ is hyperbolic. We are left with proving the conclusion of the theorem when ϑ is quasi-hyperbolic. But it is absolutely clear that an entirely similar argument can be used to prove this.

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KYUNGPOOK NATIONAL UNIVERSITY, TAEGU, KOREA